Response of Timoshenko Beams on Winkler Foundation Subjected To Dynamic Load

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Abstract - In this study, the dynamic response of a uniform deep beam resting on a Winkler elastic foundation and excited by a moving load is considered. The solution technique discussed involves the use of finite Fourier transform and the resulting simultaneous equations are reduced to simple algebraic equations via Laplace transform. Analytical and numerical solutions depict that as the values of the elastic foundation moduli increases, the amplitudes of transverse vibration of the beam decreases.

Key words- Vibration, Winkler elastic foundation, beam

INTRODUCTION

The one dimensional problem of a load moving along elastic foundation is of great technological and economical importance, as some of the results obtained may be applicable in understanding the dynamic behaviour of roadways and runways [1-3]. The dynamic response of elastic structures under the action of moving loads on elastic foundation have motivates a lot of investigations in literature [4]. Among several authors who have worked extensively in this area of study are [5] who studied the correction of shear of the differential equation for transverse vibration of prismatic bars. [6] Studied the transverse oscillations of beams under the actions of moving variable speed. [7] Considered the critical speed and the response of a tensional beam on an elastic foundation to repetitive moving loads. The effect of an added mass to the dynamic response of a prestressed beam traversed by moving masses was studied by [8]. However, in all the aforementioned works, Bernoulli-Euler beam model is often employed. Until recently, the effects of shear deformation and rotatory inertia on the dynamic response of Timoshenko beam were rarely discussed. The problem of thick beams under the action of a variable travelling transverse load was studied by [9] and in his study, he found that the transverse response of a deep beam decreases as the moving load frequency increases. [10] Studied the problem of vibration of multi-span Timoshenko beam. His study shows that the effects of rotatory inertia and shear deformation cause the modal frequencies of the Timoshenko beam to be less than those of Bernoulli-Euler beam. This paper studied the transverse displacement under moving loads of deep beams on a constant Winkler elastic foundation model. Numerical results are obtained and presented to assess the validity of the procedure adopted and to illustrate the response of the beam. Specifically, the influence of the elastic foundation on the displacement response of the beam shall be analyzed.

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DEFINITION OF THE PROBLEM

Consider a uniform Timoshenko beam of length L resting on elastic foundation with subgrade reaction moduli K, directly proportional to the beam deflection. Assuming that the beam maintains contact with the sub grade and that there is no friction forces at the interface, its deflection $V(x,t)$ from the equilibrium and rotation $\psi(x,t)$, under the action of variable magnitude moving load is described by the simultaneous second order differential equations with constant coefficient

$$\mu\frac{\partial^2 V(x,t)}{\partial t^2} - K^{\bullet}AG\left[\frac{\partial^2 V(x,t)}{\partial x^2} - \frac{\partial(n,t)}{\partial x}\right] + KV(x,t) = P(x,t) \qquad (1)$$

$$EI\frac{\partial^2 V(x,t)}{\partial x^2} + K^{\bullet}AG\left[\frac{\partial V(x,t)}{\partial x} - \psi(n,t)\right] - ID\frac{\partial^2 \psi(x,t)}{\partial t^2} = 0 \qquad (2)$$

Where x represents the traveling direction of the moving load and t represents time. Also, EI is the rigidity of the beam, E is the young's modulus of elasticity, I is the moment of inertia, A is the cross sectional area of the beam and μ is the mass per unit length of the beam. The beam density is D, G is the shear modulus, $K^{\bullet}$ is a constant depending on the shape of the cross section of the beam. The boundary conditions and initial conditions for the general beam are

$$V(x,t) = \psi_x(x,t) = 0, \text{ for } x = 0 \text{ and } l. \qquad (3a)$$

$$V(x,0) = V_t(x,0) = 0$$
$$\psi(x,0) = \psi_t(x,0) = 0 \qquad (3b)$$

$P(x,t)$ is the external load and, for a constant magnitude moving load case, can be given by

$$P(x,t) = Q\cos\omega t\,\delta(x-ut) \qquad (4)$$

Where Q is the amplitude of the applied load and δ is the Dirac-delta function, which is defined by

$$\int_{-\infty}^{\infty}\delta(x-x_0)f(x)dx = f(x_0) \quad (5)$$

When (4) is substituted into (1) and (2), the result is a non-homogeneous partial differential equations with variable coefficients. Evidently, an exact analytical solution is not feasible. Hence an approximate analytical solution is resorted to.

APPROXIMATE CLOSED FORM SOLUTION

A versatile technique termed finite Fourier sine integral transform is presented to solve the boundary-initial value problem. The technique expresses the solutions of the equations respectively as:

$$V(n,t) = \int_0^l V(x,t)\sin\frac{n\pi x}{l}dx \quad (6)$$

With inverse

$$V(x,t) = \sum_{n=1}^{\infty} V(n,t)\sin\frac{n\pi x}{l}dx \quad (7)$$

In view of the boundary conditions (3a), equation (2) is subjected to the finite Fourier cosine transform

$$\psi(n,t) = \int_0^l \psi(x,t)\cos\frac{n\pi x}{l}dx \quad (8)$$

With inverse

$$\psi(x,t) = \sum_{n=1}^{\infty} \psi(n,t)\cos\frac{n\pi x}{l}dx \quad (9)$$

Thus equations (1) and (2) become

$$\frac{\partial^2}{\partial t^2}V(n,t) - T_1\left[-\frac{n^2\pi^2}{l^2}V(n,t) - \frac{n\pi}{l}\psi(n,t) + KV(n,t)\right] = T_4\sin\frac{n\pi u}{l}t \quad (10)$$

And

$$-\frac{\partial^2}{\partial t^2}\psi(n,t) - T_3\frac{n\pi}{l}V(n,t) + \left(T_2\frac{n^2\pi^2}{l^2} + T_3\right)\psi(n,t) = 0 \quad (11)$$

Where

$$T_1 = \frac{K^{\bullet}AG}{\mu},\quad T_2 = \frac{E}{D},\quad 0\ T_3 = \frac{K^{\bullet}AG}{ID},\quad T_4 = \frac{Q}{\mu} \quad (12)$$

Subjecting equations (11) and (12) to Laplace transform

$$\Theta = \int_0^l \Theta e^{-st}dt \quad (13)$$

In conjunction with the initial conditions defined in (4), equations (10) and (11) reduced to simple algebraic simultaneous equation of the form

$$\left(s^2 + V(n,t) - T_1\frac{n^2\pi^2}{l^2} - K\right)V(n,s) - T_1\frac{n\pi}{l}\psi(n,s) = T_4\left(\frac{\omega_1}{s^2+\omega_1^2}\right) \quad (14)$$

And

$$T_3\frac{n\pi}{l}V(n,s) - \left(s^2 + T_2\frac{n^2\pi^2}{l^2} + T_3\right)\psi(n,s) = 0 \quad (15)$$

Solving equations (14) and (15) simultaneously and after some rearrangement, one obtains

$$V(n,s) = \frac{\frac{T_4 l}{T_1 n\pi}\left(s^2+\omega_F^2\right)}{\left[s^2+\left(Z_1-\sqrt{Z_2-Z_3}\right)\right]\left[s^2+\left(Z_1+\sqrt{Z_2-Z_3}\right)\right]} \quad (16)$$

And

$$\psi(n,s) = \frac{T_4\left(\frac{\omega_1}{s^2+\omega_1^2}\right)}{\left[s^2+\left(Z_1-\sqrt{Z_2-Z_3}\right)\right]\left[s^2+\left(Z_1+\sqrt{Z_2-Z_3}\right)\right]} \quad (17)$$

Equations (16) and (17) can be written as

$$V(n,s) = \frac{\frac{T_4 l}{T_1 n\pi}\left(\frac{\omega_1}{s^2+\omega_1^2}\right)\left(s^2+\omega_F^2\right)}{\left(s^2+\omega_a^2\right)\left(s^2+\omega_b^2\right)} \quad (18)$$

And

$$\psi(n,s) = \frac{T_4\left(\frac{\omega_1}{s^2+\omega_1^2}\right)}{\left(s^2+\omega_a^2\right)\left(s^2+\omega_b^2\right)} \quad (19)$$

Where

$$Z_1 = \frac{T_1 n^2\pi^2}{2l^2} - \frac{K}{2} + \frac{T_3}{2} + \frac{T_2 n^2\pi^2}{2l^2} \quad (20)$$

$$Z_2 = \left(\frac{T_1 n^2\pi^2}{2l^2} - \frac{K}{2} + \frac{T_3}{2} + \frac{T_2 n^2\pi^2}{2l^2}\right)^2 \quad (21)$$

$$Z_3 = T_1T_2\frac{n^4\pi^4}{l^4} - T_2K\frac{n^2\pi^2}{l^2} - T_3K \quad (22)$$

$$\omega_a^2 = Z_1 - \sqrt{Z_2 - Z_3} \quad (23)$$

$$\omega_b^2 = Z_1 + \sqrt{Z_2 - Z_3} \tag{24}$$

To Laplace inversion of (18), the following representation

$$f(s) = \frac{T_4 l}{T_1 n\pi}\left(\frac{\omega_1}{s^2+\omega_1^2}\right)\left(s^2+\omega_F^2\right) \tag{25}$$

$$g(s) = \frac{1}{\left(s^2+\omega_a^2\right)\left(s^2+\omega_b^2\right)} \tag{26}$$

are made, so that the Laplace inversion of (18) is the convolution of f and g. The convolution denoted by $f * g$ is represented by the integral

$$f * g = \int_0^t f(t-u)g(u)du \tag{27}$$

To this end, we obtain the inversion of (25) and (26) to obtain

$$f(t-u) = \frac{T_4 l}{T_1 n\pi}\left(\omega_F^2 \sin\omega_1(t-u) - \omega_1 \sin\omega_1(t-u)\right) \tag{28}$$

$$g(u) = \frac{1}{\left(\omega_a^2-\omega_b^2\right)}\left(\frac{\sin\omega_b u}{\omega_b} - \frac{\sin\omega_a u}{\omega_a}\right) \tag{29}$$

Thus

$$f * g = V(n,t) = \int_0^t \frac{T_4 l}{T_1 n\pi}\left(\omega_F^2 \sin\omega_1(t-u) - \omega_1 \sin\omega_1(t-u)\right)du \times \frac{1}{\left(\omega_a^2-\omega_b^2\right)}\left(\frac{\sin\omega_b u}{\omega_b} - \frac{\sin\omega_a u}{\omega_a}\right)$$

$$\begin{aligned} V(n,t) = {} & \frac{T_4\omega_F^2 L}{T_1 n\pi\omega_b\left(\omega_a^2-\omega_b^2\right)}\int_0^t \sin\omega_b \sin\omega_1(t-u)du \\ & - \frac{T_4\omega_1^2 L}{T_1 n\pi\omega_b\left(\omega_a^2-\omega_b^2\right)}\int_0^t \sin\omega_a \sin\omega_1(t-u)du \\ & - \frac{T_4\omega_F^2 L}{T_1 n\pi\omega_a\left(\omega_a^2-\omega_b^2\right)}\int_0^t \sin\omega_b \sin\omega_1(t-u)du \\ & + \frac{T_4\omega_1^2 L}{T_1 n\pi\omega_a\left(\omega_a^2-\omega_b^2\right)}\int_0^t \sin\omega_a \sin\omega_1(t-u)du \end{aligned} \tag{30}$$

Evaluating (30) after some simplifications and rearrangements yield

$$\begin{aligned} V(n,t) = {} & \frac{T_5 L_2}{2}\left(\frac{1}{z_m}\cos z_m t + \frac{1}{z_k}\cos z_k t - \frac{2\omega_b}{\omega_b^2-\omega_1^2}\right) \\ & + \frac{T_5 L_4}{2}\left(\frac{1}{z_k}\sin z_k t - \frac{1}{z_m}\sin z_m t\right) + \frac{T_5 L_2}{2}\left(\frac{1}{z_n}\cos z_n t\right. \\ & \left. + \frac{1}{z_p}\cos z_p t - \frac{2\omega_a}{\omega_a^2-\omega_1^2}\right) - \frac{T_5 L_4}{2}\left(\frac{1}{z_p}\sin z_p t - \frac{1}{z_n}\sin z_n t\right) \\ & + \frac{T_6 L_1}{2}\left(\frac{1}{z_m}\cos z_m t + \frac{1}{z_k}\cos z_k t - \frac{2\omega_b}{\omega_b^2-\omega_1^2}\right) \\ & + \frac{T_6 L_3}{2}\left(\frac{1}{z_k}\sin z_k t - \frac{1}{z_m}\sin z_m t\right) + \frac{T_6 L_1}{2}\left(\frac{1}{z_n}\cos z_n t\right. \\ & \left. + \frac{1}{z_p}\cos z_p t - \frac{2\omega_a}{\omega_a^2-\omega_1^2}\right) - \frac{T_6 L_3}{2}\left(\frac{1}{z_p}\sin z_p t - \frac{1}{z_n}\sin z_n t\right) \end{aligned} \tag{31}$$

Where

$$T_5 = \frac{T_4\omega_F^2}{T_1 n\pi}, \qquad T_6 = \frac{T_4\omega_1^2}{T_1 n\pi} \tag{32}$$

$$z_m = \omega_b + \omega_1,\ z_p = \omega_a - \omega_1,\ z_n = \omega_a + \omega_1,\ z_k = \omega_b - \omega_1 \tag{33}$$

$$\begin{aligned} L_1 &= \frac{\sin\omega_1 t}{\omega_b\left(\omega_a^2-\omega_b^2\right)},\ L_2 = \frac{\sin\omega_1 t}{\omega_a\left(\omega_a^2-\omega_b^2\right)}, \\ L_3 &= \frac{\cos\omega_1 t}{\omega_b\left(\omega_a^2-\omega_b^2\right)},\ L_4 = \frac{\cos\omega_1 t}{\omega_a\left(\omega_a^2-\omega_b^2\right)} \end{aligned} \tag{34}$$

In view of (7) the inverse transform of (31) is

$$\begin{aligned} V(x,t) = \frac{1}{l}\sum_{n=1}^{\infty}\Bigg\{ & \frac{T_5 L_2}{2}\left(\frac{1}{z_m}\cos z_m t + \frac{1}{z_k}\cos z_k t - \frac{2\omega_b}{\omega_b^2-\omega_1^2}\right) \\ & + \frac{T_5 L_4}{2}\left(\frac{1}{z_k}\sin z_k t - \frac{1}{z_m}\sin z_m t\right) + \frac{T_5 L_2}{2}\left(\frac{1}{z_n}\cos z_n t\right. \\ & \left. + \frac{1}{z_p}\cos z_p t - \frac{2\omega_a}{\omega_a^2-\omega_1^2}\right) - \frac{T_5 L_4}{2}\left(\frac{1}{z_p}\sin z_p t - \frac{1}{z_n}\sin z_n t\right) \\ & + \frac{T_6 L_1}{2}\left(\frac{1}{z_m}\cos z_m t + \frac{1}{z_k}\cos z_k t - \frac{2\omega_b}{\omega_b^2-\omega_1^2}\right) + \frac{T_6 L_3}{2} \\ & \left(\frac{1}{z_k}\sin z_k t - \frac{1}{z_m}\sin z_m t\right) + \frac{T_6 L_1}{2}\left(\frac{1}{z_n}\cos z_n t + \frac{1}{z_p}\cos z_p t\right. \\ & \left. - \frac{2\omega_a}{\omega_a^2-\omega_1^2}\right) - \frac{T_6 L_3}{2}\left(\frac{1}{z_p}\sin z_p t - \frac{1}{z_n}\sin z_n t\right)\Bigg\}\sin\frac{n\pi u}{l} \end{aligned} \tag{35}$$

Equation (35) represents the transverse displacement response to a simply supported moving load of a uniform deep beam. Similarly, taking the convolution of (19) and after some simplifications and rearrangements, it is straightforward to obtain the Laplace inversion as

$$\psi(x,t)=\frac{1}{l}\sum_{n=1}^{\infty}\frac{T_4L_1}{2}\left(\frac{1}{z_m}\cos z_m t+\frac{1}{z_k}\cos z_k t-\frac{2\omega_b}{\omega_b^2-\omega_1^2}\right)+\frac{T_4L_3}{2}\left(\frac{1}{z_k}\sin z_k t-\frac{1}{z_m}\sin z_m t\right)+\frac{T_4L_2}{2}\left(\frac{1}{z_n}\cos z_n t+\frac{1}{z_p}\cos z_p t-\frac{2\omega_a}{\omega_a^2-\omega_1^2}\right)-\frac{T_4L_4}{2}\left(\frac{1}{z_p}\sin z_p t-\frac{1}{z_n}\sin z_n t\right)\Bigg\}\cos\frac{n\pi u}{l} \quad (36)$$

NUMERICAL RESULTS

In order to illustrate the foregoing analysis, the uniform beam of length 12.20m is considered. The velocity of the moving load is taken to be 8.128metres/second. The values of foundation modulus are varied between $0N/m^3$ and $10000N/m^3$ and $\omega_1 = 2.094$. Figure 1 display the deflection profile of an elastic deep beam resting on elastic foundation and subjected to variable magnitude moving load. The figure shows that the value of foundation moduli K increases the deflection of the beam as various time t decreases.

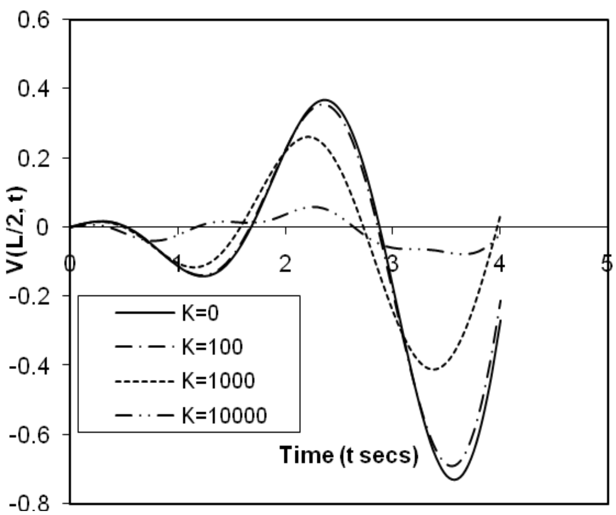

Fig. 1: Transverse displacement of a simply supported moving load for various values of foundation modulus

CONCLUSION

In this paper the response of a uniform Timoshenko beam of finite length place on winkler elastic foundation and subjected to dynamic load was studied. For the analytical solution of the governing simultaneous partial differential equations, finite Fourier transform was used with the Laplace transform. The analysis revealed that as the foundation moduli increases, the transverse deflection of the beam model decreases.

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