Global Stability Of Two Non-Linear Epidemic Model With Saturated Rate

LAID Chaharazed

Abstract: .In this work, we consider the first nonlinear epidemic model with temporary immunity and saturated incidence rate and the second model witch is modified with the new parameters. Whenever infected and recovered people will return to the susceptible class after a period. Motivated by the references of this work, which deals with a model with temporary immunity and incidence rate, means that the recovered individual has temporary immunity against a disease and he enters the sensitive class after a certain time. We present:

1. Equilibrium and stability of the disease-free equilibrium and endemic
2. Existence of endemic equilibrium and its stability analysis.
3. Global stability of the disease-free equilibrium.
4. Modified model.
5. Global stability of the disease-free equilibrium of the modified model.

The study of its sections are justified with theorems and demonstrations under certain conditions.

Index Terms: Endemic equilibrium, epidemic model, global stability, lyapunov function, stability, saturated Rate, temporary immunity,

1 INTRODUCTION

THIS paper considers the following epidemic model with temporary immunity and saturated rate:

$$\begin{cases} \dot{S}(t) = \mu - (\mu_1 + d)S(t) - (\beta + k)\frac{S(t)I(t)}{1+aI^2(t)} \\ + (\gamma_1 e^{-\mu_2 \tau} + \gamma_2 e^{-\mu_4 \tau})I(t-\tau), \\ \dot{I}(t) = (\beta + k)\frac{S(t)I(t)}{1+aI^2(t)} - (\mu_2 + d + \delta)I(t) \\ -\gamma_1 e^{-\mu_2 \tau} I(t-\tau), \\ \dot{Q}(t) = \delta I(t) - (\mu_3 + d + \alpha)Q(t), \\ \dot{R}(t) = \alpha Q(t) - (\mu_4 + d)R(t) - \gamma_2 e^{-\mu_4 \tau} I(t-\tau). \end{cases} \quad (1)$$

Consider a population of size N(t) at time t, this population is divided into for subclasses, with;

$$N(t) = S(t) + I(t) + Q(t) + R(t).$$

Where $S(t)$, $I(t)$, $Q(t)$ and $R(t)$ denote the sizes of the population; susceptible to disease, infectious members, quarantine members and recovered respectively.

The positive constants μ represent rate of incidence.

The positive constant β is the average numbers of contacts infective for S to I. The positive constant k the rate of unknown persons infected with are detected by the system. The positive constant δ is the average numbers of contacts infective for I to Q. The positive constant α is the average numbers of contacts infective for Q to R.

The positive constants μ_1, μ_2, μ_3 and μ_4 represent the death rates of susceptible, infectious and quarantine. Biologically, it is natural to assume that

$$\mu_1 \leq min\{\mu_2, \mu_3, \mu_4\}.$$

The positive constant d is natural mortality rate.

The positive constant $\gamma_i, i = 1,2$ represent the recovery rate of infection and recovery to susceptible.

- *LAID Chaharazed:- Department of Mathematics, Faculty of Exact Sciences, University Freres Mentouri. Constantine 1, Algeria. Email address: chahrazed2009@yahoo.fr. laid-chahrazed@umc.edu.dz*

The term $\gamma_1 e^{-\mu_2 \tau} I(t-\tau)$ reflects the fact that an individual has recovered from infection and still are alive after infectious period τ before becoming susceptible.

The term $\gamma_2 e^{-\mu_4 \tau} I(t-\tau)$ reflects the fact that an individual has recovered from recovered and still are alive after infectious period τ before becoming susceptible, where τ is the length of immunity period.

The formulation of the incidence rate $\frac{S(t)I(t)}{1+aI^2(t)}$; which a is saturated rate with the susceptible.

The initial condition of (1) given as:

$$\begin{cases} S(\eta) = \Phi_1(\eta),\ I(\eta) = \Phi_2(\eta), Q(\eta) = \Phi_3(\eta), \\ R(\eta) = \Phi_4(\eta), -\tau \leq \eta < 0. \end{cases} \quad (2)$$

Where $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)^T \in \square$ such that;

$$\begin{cases} S(\eta) = \Phi_1(\eta) = \Phi_1(0) \geq 0, I(\eta) = \Phi_2(\eta) = \Phi_2(0) \geq 0, \\ Q(\eta) = \Phi_3(\eta) = \Phi_3(0) \geq 0, R(\eta) = \Phi_4(\eta) = \Phi_4(0) \geq 0. \end{cases} \quad (3)$$

Let $\square$ denote the Banach space $\square([-\tau, 0], \square^4)$ of continuous functions mapping the interval $[-\tau, 0]$ into $\square^4$.

With a biological meaning, we further assume that

$$\Phi_i(\eta) = \Phi_i(0) \geq 0 \ for, i = 1,2,3,4.$$

Since $Q(t)$, $R(t)$ does not appear explicitly in the first and second equations to the system (1), in this case we study the following reduced system:

$$\begin{cases} \dot{S}(t) = \mu - (\mu_1 + d)S(t) - (\beta + k)\frac{S(t)I(t)}{1+aI^2(t)} \\ + (\gamma_1 e^{-\mu_2 \tau} + \gamma_2 e^{-\mu_4 \tau})I(t-\tau), \\ \dot{I}(t) = (\beta + k)\frac{S(t)I(t)}{1+aI^2(t)} - (\mu_2 + d + \delta)I(t) \\ -\gamma_1 e^{-\mu_2 \tau} I(t-\tau), \end{cases} \quad (4)$$

where;

$$\Phi_i(0) \geq 0, -\tau \leq \eta < 0; \ for" i = 1,2". \quad (5)$$

The region Ω

$\Omega = \{(S,I) \in \mathbb{R}_+^2,\ S+I \le N < \frac{\mu}{\mu_1+d}\}$ is positively invariant set of (3).

2 EQUILIBRIUM AND STABILITY OF THE DISEASE-FREE EQUILIBRIUM

An equilibrium point of system (4), with condition (5) satisfies,

$$\begin{cases} \dot{S}(t) = \mu - (\mu_1 + d) S(t) - (\beta + k) \dfrac{S(t) I(t)}{1 + aI^2(t)} \\ + \left(\gamma_1 e^{-\mu_2 \tau} + \gamma_2 e^{-\mu_4 \tau}\right) I(t-\tau) = 0, \\ \dot{I}(t) = (\beta + k) \dfrac{S(t) I(t)}{1 + aI^2(t)} - (\mu_2 + d + \delta) I(t) \\ -\gamma_1 e^{-\mu_2 \tau} I(t-\tau) = 0, \end{cases} \quad (6)$$

We calculate the points of equilibrium in the absence and presence of infection.

- In the absence of infection $I = 0$, the system (6) has a disease-free equilibrium E_0.

$$E_0 = \left(\hat{S}, \hat{I}\right)^T = \left(\frac{\mu}{\mu_1 + d}, 0\right)^T \quad (7)$$

Theorem 2.1. The disease-free equilibrium E_0 of the system (4) is locally asymptotically stable if $R_0 < 1$.

Proof. The eigenvalues can be determined by solving the characteristic equation of the linearization of (6) near E_0.

Therefore, the eigenvalues are

$$A_1 = -(\mu + d),$$

$$A_2 = \frac{\mu(\beta + k)}{\mu_1 + d} - f_1(\tau),\ f_1(\tau) = \mu_2 + d + \delta + \gamma_1 e^{-\mu_2 \tau}$$

In order to A_2, will be negative, then we define the basic reproduction number of the infection R_0 as follows:

$$R_0 = \frac{\mu(\beta + k)}{\mu_1 + d} \times \frac{1}{f_1(\tau)} \quad (8)$$

If $R_0 < 1$, $A_2 < 0$.

We have $A_1 < 0$, $A_2 < 0$, if $R_0 < 1$.

Then E_0 of the system (6) is locally asymptotically stable.□

3 EXISTENCE OF ENDEMIC EQUILIBRIUM AND ITS STABILITY ANALYSIS

The disease-free equilibrium E_0 of the system (4) is globally asymptotically stable if $R_0 < 1$.

- In the presence of infection $I \neq 0$; If $R_0 > 1$, the system (4) has a unique non-trivial equilibrium $E_\tau^* = (S_\tau^*,\ I_\tau^*)^T$; where

$$\begin{cases} S_\tau^* = \frac{\mu\left(1 + aI^2\right)}{\mu_1 + d} \times \frac{1}{R_0}, \\ I_\tau^* = \frac{\gamma_2 e^{-\mu_4 \tau} - (\mu_2 + d + \delta) + \sqrt{\triangle}}{2a\mu\left(1 - \frac{1}{R_0}\right)}, \\ \triangle = \left(\mu_2 + d + \delta - \gamma_2 e^{-\mu_4 \tau}\right)^2 - \left(2\sqrt{a}\mu\left(1 - \frac{1}{R_0}\right)\right)^2 \end{cases} \quad (9)$$

By introducing $x(t) = S(t) - S_\tau^*; y(t) = I(t) - I_\tau^*$, system (4) is centered at E_τ^*; we obtain the follow system:

$$\begin{cases} \dot{x}(t) = \left[-(\mu + d) - \frac{(\beta+k)}{1+aI^2} I_\tau^*\right] x + \left[\gamma_2 e^{-\mu_4 \tau} - (\mu_2 + d + \delta)\right] y \\ \dot{y}(t) = \left[\frac{(\beta+k)}{1+aI^2} I_\tau^*\right] x \end{cases} \quad (10)$$

The characteristic equation of (4) at E_τ^* is:

$$A^2 + \left[\mu + d + \frac{(\beta+k)}{1+aI^2} I_\tau^*\right] A + \frac{(\beta+k)(\mu_2 + d + \delta)}{1+aI^2} I_\tau^* - \frac{(\beta+k)(\gamma_2 e^{-\mu_4 \tau})}{1+aI^2} I_\tau^* \quad (11)$$

We introduce the following notations:

$$\begin{cases} a(\tau) = \mu + d + \frac{(\beta+k)}{1+aI^2} I_\tau^* \\ b(\tau) = \frac{(\beta+k)(\mu_2+d+\delta)}{1+aI^2} I_\tau^*, \end{cases} \quad (12)$$

With the notations in (11), the (10) becomes;

$$A^2 + a(\tau) A + b(\tau) - \frac{(\beta + k)\gamma_2}{1 + aI^2} I_\tau^* e^{-(A+\mu_4)\tau} = 0 \quad (13)$$

1. For $\tau = 0$, (5) is written as follows;

$$\begin{cases} \mu - (\mu_1 + d) S - (\beta + k) \frac{S_0^* I_0^*}{1+aI^2} + \left(\gamma_1 e^{-\mu_2 \tau} + \gamma_2 e^{-\mu_4 \tau}\right) I_0^* = 0 \\ (\beta + k) \frac{S_0^* I_0^*}{1+aI^2} - (\mu_2 + d + \delta + \gamma_1 e^{-\mu_2 \tau}) I_0^* = 0. \end{cases} \quad (14)$$

The solution of (13) is $E_0^* = (S_0^*,\ I_0^*)$. From (12) it follows, that this state is locally asymptotically stable if;

$$\mu + d + \frac{(\beta + k)}{1 + aI^2} I_\tau^* > 0 \quad (15)$$

2. For $\tau > 0$; we assume that $A = iw$; with $w > 0$ is a root of Eq (12).

Substituting into Eq (12), we obtain:

$$-w^2 + ia(\tau) w + b(\tau) - \frac{(\beta + k)\gamma_2}{1 + aI^2} I_\tau^* e^{-(A+\mu_4)\tau} = 0 \quad (16)$$

By spearing into real and imaginary parts, we obtain

$$\begin{cases} -w^2 + b(\tau) = \left(\frac{(\beta+k)\gamma_2}{1+aI^2} I_\tau^* e^{-\mu_4 \tau}\right) cos(w\tau) \\ a(\tau) w = \left(\frac{(\beta+k)\gamma_2}{1+aI^2} I_\tau^* e^{-\mu_4 \tau}\right) sin(w\tau) \end{cases} \quad (17)$$

Thus, upon squaring and adding the two equation of (16); we obtain:

$$w^4 + \left(a^2(\tau) - 2b(\tau)\right) w^2 + b^2(\tau) - \left(\frac{(\beta + k)\gamma_2}{1 + aI^2} I_\tau^* e^{-\mu_4 \tau}\right)^2 = 0 \quad (18)$$

We denote

$$\begin{cases} g_1(\tau) = a^2(\tau) - 2b(\tau) \\ g_2(\tau) = b^2(\tau) - \left(\frac{(\beta+k)\gamma_2}{1+aI^2} I_\tau^* e^{-\mu_4 \tau}\right)^2 \\ g_3(\tau) = (g_1(\tau))^2 - 4g_2(\tau) \end{cases} \quad (19)$$

With (11) and (18) we obtain;

$$\begin{cases} g_1(\tau) = (\mu + d)\left[(\mu + d) + 2\frac{(\beta+k)}{1+aI^2}\right] + \left(\frac{(\beta+k)}{1+aI^2} I_\tau^*\right)^2 \left(1 - 2(\mu_2 + d + \delta)^2\right) \\ g_2(\tau) = \left(\frac{(\beta+k)}{1+aI^2} I_\tau^*\right)^2 \left((\mu_2 + d + \delta)^2 - (\gamma_2 e^{-\mu_4 \tau})^2\right) \end{cases} \quad (20)$$

Theorem 3.1. Suppose that the conditions

$$\begin{cases} g_1(\tau) > 0, \\ g_2(\tau) > 0, \end{cases}$$

hold for all $\tau \geq 0$.

With $\frac{(\mu+d)(\beta+k)}{1+aI^2} < 0$; for all $\tau \geq 0$ $g_3(\tau)$ is also positive.

$E_\tau^* = (S_\tau^*,\ I_\tau^*)^T$, is locally asymptotically stable for all $\tau \geq 0$.

4 GLOBAL STABILITY OF THE DISEASE-FREE EQUILIBRIUM

An equilibrium point of system (1), with condition (2) satisfies,

$$\begin{cases} \mu - (\mu_1 + d) S - (\beta + k) \frac{SI}{1+aI^2} + (\gamma_1 e^{-\mu_2 \tau} + \gamma_2 e^{-\mu_4 \tau}) I(t-\tau) = 0 \\ (\beta + k) \frac{SI}{1+aI^2} - (\mu_2 + d + \delta) I - \gamma_1 e^{-\mu_2 \tau} I(t-\tau) = 0, \\ \delta I - (\mu_3 + d + \alpha) Q = 0, \\ \alpha Q - (\mu_4 + d) R - \gamma_2 e^{-\mu_4 \tau} I(t-\tau) = 0. \end{cases} \quad (21)$$

We calculate the points of equilibrium in the absence and presence of infection.

- In the absence of infection $I = 0$, the system (21) has a disease-free equilibrium $(E_0)_1$.

$$(E_0)_1 = \left(\hat{S}, \hat{I}, \hat{Q}, \hat{R}\right)^T = \left(\frac{\mu}{\mu_1 + d}, 0, 0, 0\right)^T \quad (22)$$

Theorem 4.1. The disease-free equilibrium $(E_0)_1$ of the system (1) is locally asymptotically stable if $R_0 < 1$.

Proof. The eigenvalues can be determined by solving the characteristic equation of the linearization of (1) near $(E_0)_1$.

$$\begin{cases} A_1 = -(\mu_1 + d), \\ A_2 = \frac{\mu(\beta+k)}{\mu_1+d} - f_1(\tau), f_1(\tau) = \mu_2 + d + \delta + \gamma_1 e^{-\mu_2 \tau} \\ A_3 = -(\mu_3 + d + \alpha) \\ A_4 = -(\mu_4 + d) \end{cases}$$

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In order to A_2, will be negative, then we define the basic reproduction number of the infection R_0 as follows:

$$R_0 = \frac{\mu(\beta + k)}{\mu_1 + d} \times \frac{1}{f_1(\tau)} \quad (23)$$

If $R_0 < 1$, $A_2 < 0$.

We have $A_1 < 0$, $A_2 < 0$, if $R_0 < 1$.

Then $(E_0)_1$ of the system (1) is locally asymptotically stable.

- In the presence of infection $I \neq 0$, substituting in the system, Ω also contains a unique positive, endemic equilibrium $(E_\tau^*)_1 = (S_\tau^*, I_\tau^*, Q_\tau^*, R_\tau^*)^T$.

  Where

$$\begin{cases} S_\tau^* = (1 + aI^2)\left(\frac{1}{R_0} \times \frac{\mu}{\mu_1 + d}\right), \\ I_\tau^* = \frac{R_0}{2a\mu}\left[-(f_1(\tau) + \gamma_1 e^{-\mu_2 \tau} + \gamma_2 e^{-\mu_4 \tau}) + \sqrt{\Delta}\right] \\ \Delta = (f_1(\tau) + \gamma_1 e^{-\mu_2 \tau} + \gamma_2 e^{-\mu_4 \tau})^2 + 4\left(\frac{a\mu^2}{R_0}\right): \\ Q_\tau^* = \frac{\delta}{\mu_3 + d + \alpha} \times I_\tau^*, \\ R_\tau^* = \frac{1}{\mu_4 + d}\left[\frac{\alpha\delta}{\mu_3 + d + \alpha} - \gamma_2 e^{-\mu_4 \tau}\right] \times I_\tau^* \end{cases} \quad (24)$$

Theorem 4.2 The disease-free equilibrium $(E_0)_1$ of the system (1) is locally asymptotically stable on Ω if $R_0 \leq 1$.

With $R_0 > 1$, the system (1) has a unique non-trivial equilibrium $(E_\tau^*)_1$, which is globally asymptotically stable.

Proof. Choose the lyapunov functional $V: \Omega_1 \to R$.

$$\Omega_1 = \left\{ \begin{matrix} S(t), I(t), Q(t), R(t) \in \Omega, \\ S(t) > 0, I(t) > 0, Q(t) > 0, R(t) > 0 \end{matrix} \right\} \quad (25)$$

$$V(S, I, Q, R) = \begin{cases} w_1\left[(S - S^*) ln \frac{S}{S^*}\right] + w_2\left[(I - I^*) ln \frac{I}{I^*}\right] + \\ w_3\left[(Q - Q^*) ln \frac{Q}{Q^*}\right] + w_4\left[(R - R^*) ln \frac{R}{R^*}\right] \end{cases} \quad (26)$$

Where $w_i, i = 1, 2, 3, 4$ are positive.

The derivative of (25) is;

$$\dot{V}(S, I, Q, R) = \begin{cases} w_1\left[(S - S^*)\left(\frac{\mu}{S} + \frac{I}{S}\left(\gamma_1 e^{-\mu_2 \tau} + \gamma_2 e^{-\mu_4 \tau}\right) - (\beta + k)\frac{I}{1+aI^2} - (\mu_1 + d)\right)\right] \\ w_2\left[(I - I^*)\left((\beta + k)\frac{S}{1+aI^2} - (\delta + \gamma_1 e^{-\mu_2 \tau}) - (\mu_2 + d)\right)\right] \\ + w_3\left[(Q - Q^*)\left(\delta \frac{I}{Q} - \alpha - (\mu_3 + d)\right)\right] \\ + w_4\left[(R - R^*)\left(\alpha \frac{Q}{R} - \gamma_2 e^{-\mu_4 \tau} \frac{I}{R}\right) - (\mu_4 + d)\right] \end{cases} \quad (27)$$

Then we have,

$$\dot{V}(S, I, Q, R) = \begin{cases} -\left[\mu w_1 \frac{(S-S^*)^2}{SS^*}\right] - \left[\left(\gamma_1 e^{-\mu_2 \tau} + \gamma_2 e^{-\mu_4 \tau}\right) w_1 \frac{(S-S^*)(SI^* - IS^*)}{SS^*}\right] \\ - \left[\frac{\beta + k}{1+aI^2}(S - S^*)(I - I^*)(w_1 - w_2)\right] \\ - \left[\delta w_3 \frac{(Q-Q^*)}{QQ^*}(QI^* - IQ^*)\right] \\ - \left[w_4 \frac{(R-R^*)}{RR^*}\left(\alpha(RQ^* - QR^*) + \gamma_2 e^{-\mu_4 \tau}(IR^* - RI^*)\right) + \right] \end{cases} \quad (28)$$

For $w_i = 1, i = 1, 2, 3, 4$; (27) becomes;

$$\dot{V}(S, I, Q, R) = \begin{cases} -\left[\mu \frac{(S-S^*)^2}{SS^*}\right] - \left[\left(\gamma_1 e^{-\mu_2 \tau} + \gamma_2 e^{-\mu_4 \tau}\right) \frac{(S-S^*)(SI^* - IS^*)}{SS^*}\right] \\ - \left[\delta \frac{(Q-Q^*)}{QQ^*}(QI^* - IQ^*)\right] \\ - \left[w_4 \frac{(R-R^*)}{RR^*}\left(\alpha(RQ^* - QR^*) + \gamma_2 e^{-\mu_4 \tau}(IR^* - RI^*)\right) + \right] \end{cases} \quad (29)$$

We have

$\dot{V}(S, I, Q, R) \leq 0;\ if, S = S^*, I = I^*, Q = Q^*, R = R^*;$

with Lasalle; $(E_\tau^*)_1$ which is globally asymptotically stable.□

5 . MODIFIED MODEL

We modify model (1), with new well-defined parameters. We obtain the following model:

$$\begin{cases} \dot{S}(t) = pS(t)\left(1-\frac{S(t)+I(t)}{b}\right)+\left(\gamma_1 e^{-\mu_2\tau}+\gamma_2 e^{-\mu_4\tau}\right)I(t-\tau)-(\mu_1+d)S(t) \\ -(\beta+k)\frac{S(t)I(t)}{1+aI^2(t)}-M_1S(t), \\ \dot{I}(t) = (\beta+k)\frac{S(t)I(t)}{1+aI^2(t)}-(\mu_2+d)I(t)-\gamma_1 e^{-\mu_2\tau}I(t-\tau)-\frac{mI(t)Q(t)}{S(t)+I(t)+aQ(t)}-M_2I(t) \\ \dot{Q}(t) = \frac{mI(t)Q(t)}{S(t)+I(t)+aQ(t)}-(\mu_3+d+\alpha)Q(t), \\ \dot{R}(t) = \alpha Q(t)-(\mu_4+d)R(t)-\gamma_2 e^{-\mu_4\tau}I(t-\tau). \end{cases} \quad (3027)$$

Where M_1S, M_2I represent the catch of S and I. p represent the intrinsic growth rate. b represent the carring capacity of the environment. m represent the research rate.

The initial condition of (30) given as:

$$S(t)>0, I(t)>0, Q(t)>0, R(t)>0.$$

We apply the change of variable chosen as follows:

$$s=\frac{S}{b}, i=\frac{I}{b}, q=\frac{Q}{b}, r=\frac{R}{b}, \sigma=pt, \quad (31)$$

By applying the required changes to system (30) we obtain the new system which will be studied later, which is given as follows:

$$\begin{cases} \frac{ds}{d\sigma} = s(1-s-i)-\left(\frac{b(\beta+k)}{p(1+ab^2i^2)}\right)si-\frac{(\mu_1+d+M_1)}{p}s+\frac{\left(\gamma_1 e^{-\mu_2\tau}+\gamma_2 e^{-\mu_4\tau}\right)}{p}i, \\ \frac{di}{d\sigma} = \left(\frac{b(\beta+k)}{p(1+ab^2i^2)}\right)si-\frac{\left(\mu_2+d+M_2+\gamma_1 e^{-\mu_2\tau}\right)}{p}i-\frac{m}{ap(s+i+q)}iq, \\ \frac{dq}{d\sigma} = \left(\frac{m}{p(s+i+q)}\right)iq-\frac{(\mu_3+d+\alpha)}{ap}q, \\ \frac{dr}{d\sigma} = \frac{\alpha}{ap}q-\frac{\mu_4+d}{p}r-\frac{\gamma_2 e^{-\mu_4\tau}}{p}i. \end{cases} \quad (32)$$

5.1 Global stability of the disease-free equilibrium

An equilibrium point of system (32), satisfies,

$$\begin{cases} s(1-s-i)-\left(\frac{b(\beta+k)}{p(1+ab^2i^2)}\right)si-\frac{(\mu_1+d+M_1)}{p}s+\frac{\left(\gamma_1 e^{-\mu_2\tau}+\gamma_2 e^{-\mu_4\tau}\right)}{p}i=0 \\ \left(\frac{b(\beta+k)}{p(1+ab^2i^2)}\right)si-\frac{\left(\mu_2+d+M_2+\gamma_1 e^{-\mu_2\tau}\right)}{p}i-\frac{m}{ap(s+i+q)}iq=0, \\ \left(\frac{m}{p(s+i+q)}\right)iq-\frac{(\mu_3+d+\alpha)}{ap}q=0, \\ \frac{\alpha}{ap}q-\frac{\mu_4+d}{p}r-\frac{\gamma_2 e^{-\mu_4\tau}}{p}i=0. \end{cases} \quad (28)$$

- In the absence of infection $I=0$, the system (32) has a disease-free equilibrium $(E_0)_2$.

$$(E_0)_2=\left(\hat{s},\hat{i},\hat{q},\hat{r}\right)^T=\left(1-\frac{\mu_1+d+M_1}{p},0,0,0\right)^T \quad (34)$$

Theorem 5.1. The disease-free equilibrium $(E_0)_2$ of the system (32) is locally asymptotically stable if $R_0<1$.

Proof. The eigenvalues can be determined by solving the characteristic equation of the linearization of (32) near $(E_0)_2$.

Then we define the basic reproduction number of the infection R_0 as follows:

$$R_0=\frac{b(\beta+k)(p-(\mu_1+d+M_1))}{\mu_2+d+M_2+\gamma_1 e^{-\mu_2\tau}} \quad (35)$$

If $R_0<1$

Then $(E_0)_2$ of the system (32) is locally asymptotically stable.□

- In the presence of infection $I\neq 0$, the endemic equilibrium, $(E_\tau^*)_2=\left(s_\tau^*,i_\tau^*,q_\tau^*,r_\tau^*\right)$.

Theorem 5.2. The disease-free equilibrium $(E_0)_2$ of the system (32) is globally asymptotically stable if $R_0<1$.

Proof. Choose the Lyapunov functional

$$V(x_1,x_2,x_3,x_4)=x_1(s-\hat{s})+x_2\left(i-\hat{i}\right)+x_3(q-\hat{q})+x_4(r-\hat{r}). \quad (36)$$

The derivative $\frac{dV(x_1,x_2,x_3,x_4)}{d\sigma}$ is

$$\frac{dV(x_1,x_2,x_3,x_4)}{d\sigma}=x_1\frac{ds}{d\sigma}+x_2\frac{di}{d\sigma}+x_3\frac{dq}{d\sigma}+x_4\frac{dr}{d\sigma} \quad (37)$$

Where x_1, x_2, x_3, x_4 are positive constants

$$\begin{aligned} \frac{dV(x_1,x_2,x_3,x_4)}{d\sigma} &= x_1\begin{bmatrix} s(1-s-i)-\left(\frac{b(\beta+k)}{p(1+ab^2i^2)}\right)si \\ -\frac{(\mu_1+d+M_1)}{p}s+\frac{\left(\gamma_1 e^{-\mu_2\tau}+\gamma_2 e^{-\mu_4\tau}\right)}{p}i \end{bmatrix} \\ &+ x_2\left[\left(\frac{b(\beta+k)}{p(1+ab^2i^2)}\right)si-\frac{\left(\mu_2+d+M_2+\gamma_1 e^{-\mu_2\tau}\right)}{p}i-\frac{m}{ap(s+i+q)}iq\right] \\ &+ x_3\left[\left(\frac{m}{p(s+i+q)}\right)iq-\frac{(\mu_3+d+\alpha)}{ap}q\right] \\ &+ x_4\left[\frac{\alpha}{ap}q-\frac{\mu_4+d}{p}r-\frac{\gamma_2 e^{-\mu_4\tau}}{p}i\right] \end{aligned}$$

Then

$$\begin{aligned} \frac{dV(x_1,x_2,x_3,x_4)}{d\sigma} = & \left[s(1-s-i)\right]x_1-\left[\left(\frac{b(\beta+k)}{p(1+ab^2i^2)}\right)si\right](x_1-x_2)-\left[\frac{(\mu_1+d+M_1)}{p}s\right]x_1 \\ & -\left[\frac{\gamma_1 e^{-\mu_2\tau}}{p}i\right](x_2-x_1)-\left[\frac{\gamma_2 e^{-\mu_4\tau}}{p}i\right](x_4-x_2)-\left[\frac{(\mu_2+d+M_2)}{p}i\right]x_2 \\ & -\left[\left(\frac{m}{p(s+i+q)}\right)iq\right]\left(\frac{x_2}{a}-x_3\right)-\left[\frac{(\mu_3+d)}{ap}q\right]x_3-\left[\frac{\alpha}{ap}q\right](x_3-x_4)-\left[\frac{\mu_4+d}{p}r\right]x_4 \end{aligned} \quad (38)$$

Let choose $x_1=x_2=x_3=x_4=1$.

We obtain

$$\begin{aligned} \frac{dV(x_1,x_2,x_3,x_4)}{d\sigma} \leq & -\left[\frac{(\mu_1+d+M_1)}{p}s\right]-\left[\frac{(\mu_2+d+M_2)}{p}i\right]-\left[\left(\frac{m(1-a)}{ap(s+i+q)}\right)iq\right] \\ & -\left[\frac{(\mu_3+d)}{ap}q\right]-\left[\frac{\mu_4+d}{p}r\right]x_4. \end{aligned} \quad (39)$$

$\frac{dV(x_1,x_2,x_3,x_4)}{d\sigma}<0$, $(E_0)_2$ is globally asymptotically stable if $R_0<1$. □

6 CONCLUSION

This article studies two models of which the second is the first modifies with different parameters and under certain conditions. This paper addresses the equilibrium and stability of the disease-free equilibrium of the epidemic model restrain

with temporary immunity and saturated incidence rate, in the absence of infection, the system restrain has a disease-free equilibrium witch is locally asymptotically stable if $R_0 < 1$, then existence of endemic equilibrium and its stability analysis under some conditions. Finally, we study global stability of the disease-free equilibrium to the modified model.

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