A New Modified Version Of Gauss-Seidel Iterative Method Using Grouping Relaxation Approach

Baher A. Haleem, Ihab M. El Aghoury, Bahaa S. Tork, Hisham A. El-Arabaty

Abstract: Systems of linear equations appear in many areas either directly as in modeling physical situations or indirectly as in the numerical solutions of other mathematical models. The solution of the linear equations' system is probably the most important issue in numerical methods like the finite element method (FEM). Using the finite element method in modeling various structures, with either simple or complicated configuration of elements, in structural engineering became prevalent many years ago. There are two main types of solvers depending on whether the used method is direct or iterative (indirect) method. In contrast to the iterative techniques, the direct techniques provide almost exact solutions, however they are not convenient for some situations, including but not limited to huge systems of equations. In such situations, the iterative solvers are favored as they have privileges concerning solving speed and storage requirements. In addition, indirect solvers are simpler to program. This research focuses on using the Classical (Stationary) iterative techniques for solving linear systems of equations. The main objective of this research is to develop a new modified version of the well-known Gauss-Seidel (GS) iterative technique which is adapted to solving problems in structural engineering. The proposed technique remarkably outperforms GS technique regarding the required number of iterations and the convergence speed. In this paper, the differences between the direct and iterative approaches have been discussed, along with a quick overview of some of the methods underlying these two classes. Then, the idea and algorithm of the new proposed "Modified Gauss-Seidel" (MGS) technique have been elucidated. Afterward, the algorithm has been programmed and used to solve some 2D Practical Examples, besides using the conventional Jacobi and GS techniques. Finally, the obtained results have been compared to assess the proposed MGS; it outperformed both Jacobi and GS.

Index Terms: Systems of linear equations, Finite element method, Direct techniques, Indirect techniques, Stationary iterative methods, Jacobi, Gauss-Seidel.

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1 INTRODUCTION

Systems of linear equations appear in many areas either directly as in modeling physical situations or indirectly as in the numerical solutions of other mathematical models. These applications arise in various fields, including but not limited to engineering, biological, physical, social science, etc [1]. The solution of the linear equations' system is probably the most important issue in numerical methods. The simplest models of the physical field are linear, hence linear equations appear frequently in physical problems. Not only that, but even the most complicated cases are frequently approximated through using a linear model as an initial step. Moreover, the solution of a nonlinear equations' system is achieved by an iterative process involving the solution of a sequence of linear systems, where each of them approximates the nonlinear equations. Likewise, the solution of differential along with integral equations by discretizing the system using normally the finite difference technique leads to a system of nonlinear or linear equations [2]. The linear system problem can be expressed in matrix form such that [A]{x}={b}, where [A] is an n×n nonsingular matrix so-called coefficient matrix, {b} is an n-vector of known constants, and the problem is to find the values in the n-vector {x} which is the vector of the unknowns [1]. This linear system usually appears in numerical analysis problems in many engineering departments, as previously mentioned, however this research focuses on structural analysis problems in civil engineering only. Using the finite element method in modeling various structures, with either simple or complicated configuration of elements, in structural engineering became prevalent many years ago. In this paper, a new modified approach of Gauss-Seidel iterative technique is introduced for solving the systems of linear equations resulting from applying the stiffness method to structural models using the finite element analysis. Generally, the used techniques for solving systems of linear equations are mainly classified into Direct techniques and Iterative techniques.

1.1 Direct methods

The direct methods get the exact solution within a finite amount of computation, so long as there is not any roundoff error [2]. But in reality, the exact solution is obtained where there is no error apart from the round off error [1]. Among the direct methods are Gaussian elimination, Gauss-Jordan elimination, Cholesky decomposition, LU decomposition, etc.

1.1.1 Gaussian elimination method

The Gauss elimination method is one of the oldest direct methods where the process starts by the so-called "forward elimination" where some row operations are carried out in order to reach a triangular form of equations in the coefficient matrix leaving zeros in place of the off-diagonal entries below the main diagonal. Afterward, this is followed by the process known as "back-substitution" in which the unknown of the last row (Xn), which was obtained directly from the last equation as it has not any other unknown, is then substituted back in the previous equation to get the value of the previous unknown (Xn-1), and so on until all the values of all unknowns are attained eventually [3].

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1.1.2 Gauss-Jordan Elimination method
In the Gauss-Jordan Elimination method, only the “forward elimination” process is applied, since all the off-diagonal elements, below and above the main diagonal, are eliminated at the same time. Hence, the “back-substitution” process is no longer needed [3].

1.2 General merits and demerits of both classes of methods
Direct techniques have some difficulties. For instance, the problem of the method of Gaussian elimination lies in the accumulation of rounding errors. This has stimulated many authors to scrutinize the solutions of systems of linear equations by both direct and indirect techniques [1]. Direct techniques are almost habitually used for solving equations of filled matrices, whereas iterative techniques are sometimes favored for sparse matrices. It may not be possible to exploit sparsity of sparse matrices while using direct techniques because the elimination process can make the zero entries nonzero, which does not happen in the iterative techniques. Hence, the storage requirements and number of arithmetic operations may be the same for filled and sparse matrices which may be prohibitive for significantly large matrices. Thus, it may be worthy to resort to the iterative techniques in those cases of large systems as they are less time-consuming and require less storage on a computer. Furthermore, some matrices, usually resulting from solution of partial differential equations, may have large bandwidth besides a large fraction of zero entries within the band. The iterative techniques may be considered also for such matrices [2]. With significant advances in methods for treatment of sparse matrices, the usage of direct techniques is increasing, even in more general cases. However, the situation may be changed in favor of iterative techniques when it comes to machines that are capable of parallel processing since the iterative algorithm can be effortlessly adapted for parallel processing as the equations can be solved independently [2].

1.3 Iterative methods
The iterative methods provide approximate solutions where there is some error within the desired tolerance value. Basically, they give a sequence of approximations that converges to the final solution [1]. The iterative methods are divided into two main categories of methods: Classical (Stationary) methods and Modern methods. In this research, only two Classical methods—Jacobi and Gauss-Seidel methods—upon which the newly developed iterative technique is based, will be discussed.

1.3.1 Jacobi iterative method
The Jacobi iterative technique was devised by a German mathematician whose name is Carl Gustav Jacob Jacobi (1804–1851). The Jacobi iterative technique is also known as the method of simultaneous displacements; this is for the fact that the value of every entry of the solution (deformations) vector is changed, however before any of the new values of the entries are used in that iteration. Thus, both vectors of $\{x_i\}$ and $\{x_{i-1}\}$ need to be stored separately [2]. Iterative formula of the Jacobi procedure is given by (1); where k is the iteration's number; i and j are the row's order and column's order respectively in the coefficient matrix; n is the total number of unknowns; a, b, and x refer to the terms of the coefficient matrix, vector of known constants, and vector of unknowns respectively.

$$x_i^{k+1} = \frac{1}{a_{ii}}(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j^{\,k}), \qquad (1)$$

$$i = 1, 2, ..., n$$

Nevertheless, the Jacobi technique is seldom used in practice as the Gauss-Seidel technique is remarkably faster and is implemented easier on a sequential computer [2].

1.3.2 Gauss-Seidel (GS) iterative method
The Gauss-Seidel (GS) technique was devised by both Carl Friedrich Gauss (1777–1855) and Philipp L. Seidel (1821–1896). GS is the modification of the Jacobi iterative technique. The difference between both the Jacobi and GS techniques is that in the GS method each component of x is used immediately in the iteration as soon as it is computed. Accordingly, Gauss-Seidel technique is occasionally named the method of successive displacements. GS method is more appropriate for programming as the successive approximations do not need to be stored in two separate arrays; the new values can be overwritten immediately on the old values [2]; which in turn requires less storage on a computer and that will be significantly effective in case of huge systems, for instance when solving for millions of unknowns [4]. However, it should be noted that the GS iteration will not be really useful on a parallel computer at which all the computation of a Jacobi iteration can be performed simultaneously whilst for Gauss-Seidel any unknown cannot be computed until the preceding unknown is calculated; it is a sequential process [3].
Iterative formula of the Gauss-Seidel procedure is given by (2):

$$x_i^{k+1} = \frac{1}{a_{ii}}(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{\,k+1} - \sum_{j=i+1}^{n} a_{ij} x_j^{\,k}), \qquad (2)$$

$$i = 1, 2, ..., n$$

A comparison between Jacobi technique and GS technique was presented by A. I. Bakari and I. A. Dahiru [5]. Their results showed that GS outperforms Jacobi technique as far as the accuracy and the number of iterations till convergence are concerned; GS is faster than Jacobi. The convergence conditions of Jacobi technique and GS technique were shown by Liu Hongxia and Feng Tianxiang [6]. Davod Khojasteh Salkuyeh [7] introduced a generalization of Jacobi technique and Gauss-Seidel technique and studied their convergence properties. The numerical results of his research proved that his new generalized procedures have been more efficient than conventional Jacobi technique and Gauss-Seidel technique. It has been shown that if the coefficient matrix [A] is irreducibly diagonally dominant or if it is strictly diagonally dominant (SDD), the associated iterations of Jacobi technique and Gauss-Seidel technique converge for any initial guess [8]. Also, if the coefficient matrix is symmetric positive definite (SPD), the Gauss-Seidel technique also converges for any initial guess x_o [9]. Moreover, the matrix is considered strictly diagonally dominant providing the condition presented by (3) is valid [10]:

$$\sum_{j \neq i} |a_{ij}| < |a_{ii}| \quad (3)$$

2 DEVELOPMENT OF THE MODIFIED GAUSS-SEIDEL (MGS) TECHNIQUE

The Modified Gauss-Seidel (MGS) method is an iterative technique adapted for solving structural problems. MGS is based on the well-known Gauss-Seidel method which is considered as an iterative method among the Classical (Stationary) techniques. What makes the MGS method different from the GS method is the addition of a certain relaxation mechanism that boosts the performance of this new method and makes it remarkably better than GS method. In fact, the MGS method includes some matrix inversion processes upon which the relaxation mechanism depends. However, the inverted matrices are significantly smaller than the global stiffness matrix of the entire structure. Thus, the disadvantages of the direct solvers, represented in matrix inversion process or matrix decomposition, are greatly alleviated, in addition to exploiting the merits of the iterative methods.

2.1 Beam theories

There are two main beam theories which are Euler-Bernoulli beam theory and Timoshenko beam theory. Timoshenko beam theory accounts for shear deformation effects, as such it gives more accurate results. So, Timoshenko beam theory is adopted in this research.

2.2 2D Sign convention

The physical significance of the signs in 2D that is used in this research for the loads (forces and moments) and deformations (displacements and rotations) are shown in Fig. 1.

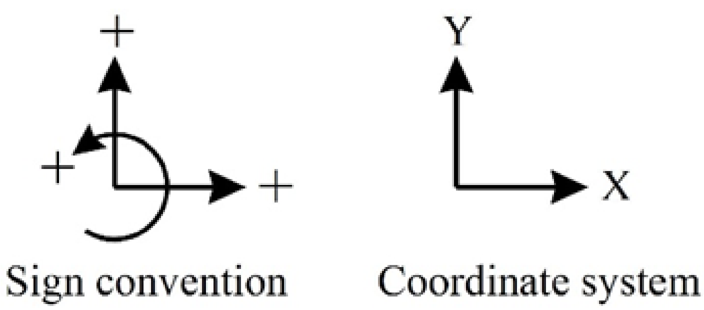

Fig. 1*. The considered positive directions in 2D*

2.3 The algorithm of MGS

The algorithm of MGS is divided into two iteration cycles that are distinct from each other. They are repeated one after another until convergence is realized. The first iteration cycle follows the same concept of Gauss-Seidel iteration cycle. However, the other cycle is the innovative one where the developed relaxation mechanism is applied to certain degrees of freedom (DOFs), while the other DOFs are relaxed following the idea of the conventional GS method. The Example shown in Fig. 2 will be adopted to illustrate the idea of MGS technique. This Example is a 2D frame of single bay and three stories with fixation supports. The nodes are numbered as shown in Fig. 2. This 2D Example has 18 DOFs, where 6 nodes * 3 DOFs per node = 18 DOFs.

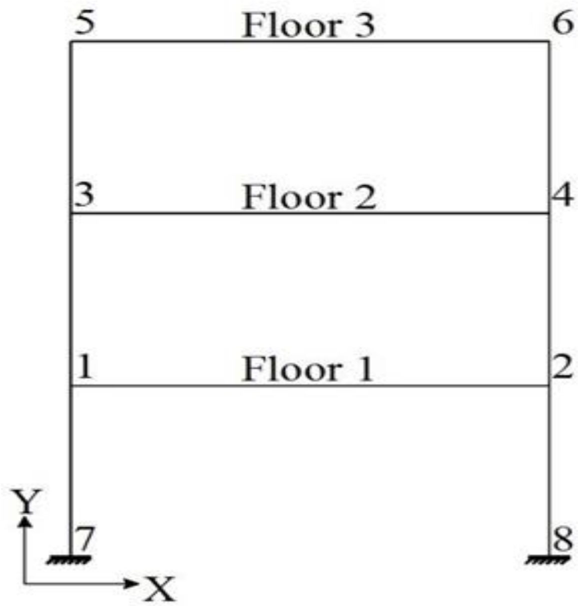

Fig. 2*. A 2D frame of a single bay and three stories*

As previously mentioned, the MGS consists of two different iteration cycles that are repeated mutually in a successive manner till convergence. In this research, the developed technique is applied to solve the system of linear equations resulting from the finite element analysis in matrix form. Accordingly, the MGS technique will be illustrated in the subsequent sections in matrix form to be easily used by computers as a program for solving those systems of equations.

2.3.1 The standard iteration cycle (Cycle of uncoupled DOFs)

In this iteration cycle, the concept of the conventional GS method is applied. If this cycle were to follow the Jacobi iterative procedure, then the vector of the residual loads would be updated once at the beginning of each iteration cycle through this calculation: $\{R\}_{18x1} = \{F_0\}_{18x1} - [K]_{18x18} * \{\Delta\}_{18x1}$, where {R} is the vector of the residual nodal loads, $\{F_0\}$ is the vector of the initially applied nodal loads, [K] is the global stiffness matrix of the entire structure, {Δ} is the vector of nodal deformations. Afterward, the residual load value of each DOF would be divided by the corresponding diagonal term in the stiffness matrix to get the change in the value of the deformation of this DOF in this iteration. Then, these changes in the deformation values are added to the previous deformation values to get the new values. The idealized stiffness matrix shown in Fig. 3 describes this relaxation procedure where only the diagonal terms are involved, whilst the other stiffness terms are assumed to be zeros temporarily in this step. Nevertheless, this iteration cycle follows the Gauss-Seidel concept, hence the value of the residual load (whether force or moment) of each DOF is updated individually just before the calculation of the corresponding deformation using this residual value. All DOFs are considered in this manner in sequence, then this iteration is finished.

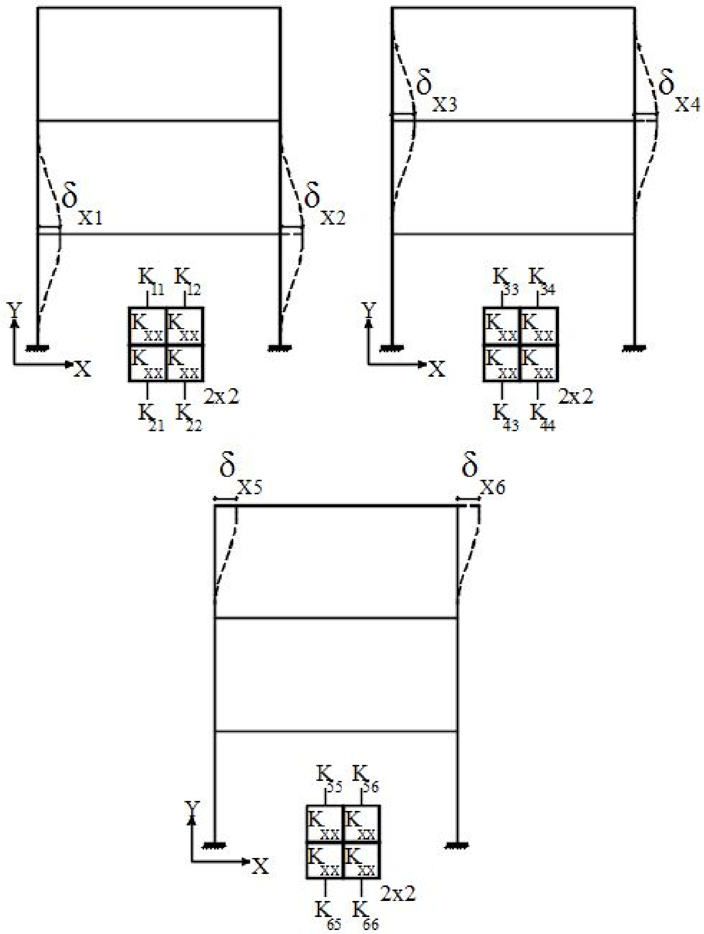

***Fig. 3**. Idealized stiffness matrix of the standard iteration cycle*

2.3.2 The modified iteration cycle (Cycle of coupled DOFs)

In this iteration cycle, a new relaxation mechanism is adopted as some DOFs are chosen to be coupled (grouped) together during the relaxation process, while the other DOFs follow the same concept of relaxation adopted in the standard iteration cycle. It has been found through this research that coupling the lateral displacements of the nodes (δ_X) on the same floor to be relaxed together at the same instant results in faster convergence. This mechanism is achieved through extracting an idealized submatrix for each floor from the global stiffness matrix of the entire structure. Each submatrix is inverted once at the beginning of the analysis process to be multiplied by the subvector of the residual lateral forces of the corresponding nodes every modified iteration cycle to get the change in the values of these nodal lateral displacements, where each subvector is updated just before this multiplication in each modified iteration cycle. Then, these changes in the deformation values are added to the previous deformation values to get the new values. However, this matrix inversion process is not as exhausting as inverting the global stiffness matrix of the structure since the order of each submatrix equals the number of nodes in this floor. Hence, in this Example there are 3 submatrices each of order 2*2, however the order of the global stiffness matrix is 18*18. Moreover, the direct techniques are generally preferred to iterative techniques providing the matrix is banded with small bandwidth [2], which is the case in those submatrices. Fig. 4 shows the relaxation mechanism of lateral displacements only in this modified iteration cycle for the 1st, 2nd, and 3rd floors. Besides, the extracted submatrices of the three floors are also provided in Fig. 4.

	Node 1			Node 2			Node 3			Node 4			Node 5			Node 6		
	K_{XX}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Node 1	0	K_{YY}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	K_{ZZ}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	K_{XX}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Node 2	0	0	0	0	K_{YY}	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	K_{ZZ}	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	K_{XX}	0	0	0	0	0	0	0	0	0	0	0
Node 3	0	0	0	0	0	0	0	K_{YY}	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	K_{ZZ}	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	K_{XX}	0	0	0	0	0	0	0	0
Node 4	0	0	0	0	0	0	0	0	0	0	K_{YY}	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	K_{ZZ}	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	K_{XX}	0	0	0	0	0
Node 5	0	0	0	0	0	0	0	0	0	0	0	0	0	K_{YY}	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	K_{ZZ}	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	K_{XX}	0	0
Node 6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	K_{YY}	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	K_{ZZ}

***Fig. 4**. Grouping of lateral displacements*

The other DOFs, nodal vertical displacements (δ_Y) and nodal rotation about Z-axis (Θ_Z), are relaxed individually as in the standard iteration cycle.

3 ASSESSMENT OF THE PERFORMANCE OF MGS IN 2D ANALYSIS

In this section, the newly developed Modified Gauss-Seidel (MGS) iterative technique has been compared to the conventional Jacobi and Gaus-Seidel (GS) iterative methods to appraise the efficiency of the developed procedure. The Jacobi, GS, and MGS techniques have been programmed for this research work using the Visual Basic for Applications (VBA) programming language through the Microsoft Excel 2016 program on a Dell Inspiron N5110, Core i7 laptop.

3.1 Convergence criterion and tolerances

The convergence is attained when all the values of the residual loads are within the tolerance range. For instance, a tolerance value of 0.001 means that all the residual loads' values should be reduced gradually with iterations until they lie within the range of -0.001 to +0.001 for the solver to stop going into further iterations. It has been found through this research that a tolerance value of 0.001 is sufficient to obtain deformation values with almost the same accuracy as that of the results obtained from SAP2000 commercial software.

3.2 2D Examples for the performance appraisal of MGS

In this section, some Practical Examples of 2D single-story and multi-story buildings are solved using Jacobi, Gauss-Seidel (GS), and Modified Gauss-Seidel (MGS) iterative techniques. The objective of those Examples is to compare the results—the number of iterations and elapsed time in analysis—of these techniques in order to assess the performance of MGS and to show how it greatly outperformed the GS method. All the Examples have the same material properties. The considered material is Reinforced Concrete with Young's modulus (E) = $22*10^5$ t/m^2 and with Poisson's ratio (υ) = 0.2, whilst the shear modulus (G) depends on both E and υ such that G = E / [2 * (1 + υ)]. In any Example, the entire frame structure is simulated (modeled) using only beam elements (frame elements)—just for simplicity. The cross sections of the beams and columns are rectangular. The shear area coefficient of these rectangular sections is taken equal to 6/5 [11]. The beams and columns are considered massless; all the loads are externally applied to the nodes. The deformation values are initially set to zeros as an initial guess. The beams' and columns' lengths, dimensions of rectangular cross sections, and applied nodal loads of all the considered Examples are as follows:

Beams: Span = 10 m, Width = 0.3 m, and Depth = 1 m

Columns: Height = 5 m, Width = 0.3 m, and Depth = 0.9 m

Nodal loads: F_{Xo} = 15 tonf, F_{Yo} = -70 tonf, and M_{Zo} = -100 mtonf

Where the applied nodal nodes are applied to all nodes except that F_{Xo} is applied to the nodes at the left edge only of any building, as shown in Fig. 5 and Fig. 6. It should be noted that it has been tried through this research to use the developed MGS technique however without the inclusion of the standard iteration cycle; only the modified iteration cycle has been repeated successively till convergence. This form of MGS has been given the acronym of MGS* as shown in Table 1 and Table 2. Table 1 and Table 2 show the number of DOFs of nine 2D Practical Examples, in addition to the results (Number of iterations and analysis time) using Jacobi, GS, MGS, and MGS* methods. However, the Jacobi method was used in the first three Examples only because it is already known from the previous researches in the literature that Jacobi is slower than GS method, but it was inserted here in order to just emphasize this fact (Fig. 7). Fig. 5 and Fig. 6 show the configuration and number of bays and floors of the nine Examples mentioned in Table 1 and Table 2. Fig. 7 is a bar chart that shows the analysis time in seconds using both Jacobi and GS methods for Examples 1 to 3. Fig. 8 is a bar chart that shows the analysis time in seconds using both GS and MGS methods for Examples 1 to 9. Fig. 9 is a bar chart that shows the analysis time in seconds using both MGS and MGS* methods for Examples 1 to 9.

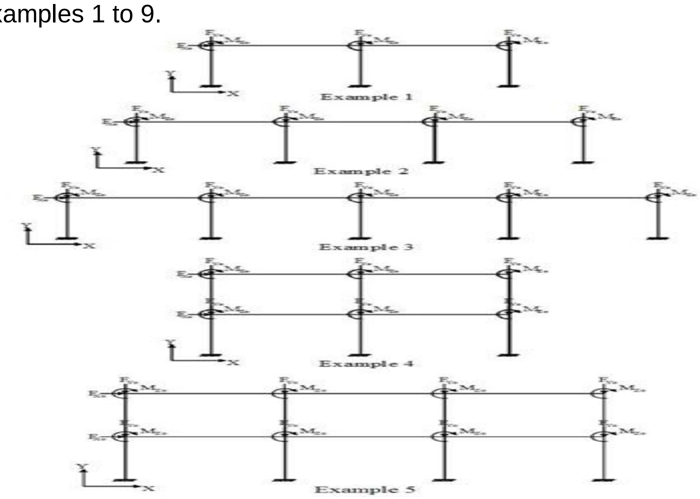

Fig. 5. *The configuration of the members of Examples 1 to 5*

It can be noticed from Table 1 and Table 2 that although some Examples with a relatively large number of DOFs have a smaller number of iterations than other Examples with a relatively small number of DOFs, the latter Examples have shorter analysis time. This is because the elapsed time of the single iteration increases by increasing the number of unknowns. By comparing Example 4 with 6, it is obvious that Example 6 has a larger number of iterations and analysis time, whether using GS or MGS, although both Examples have the same number of DOFs. This indicates that increasing the number of floors has a worse effect on the solving speed than increasing the number of bays.

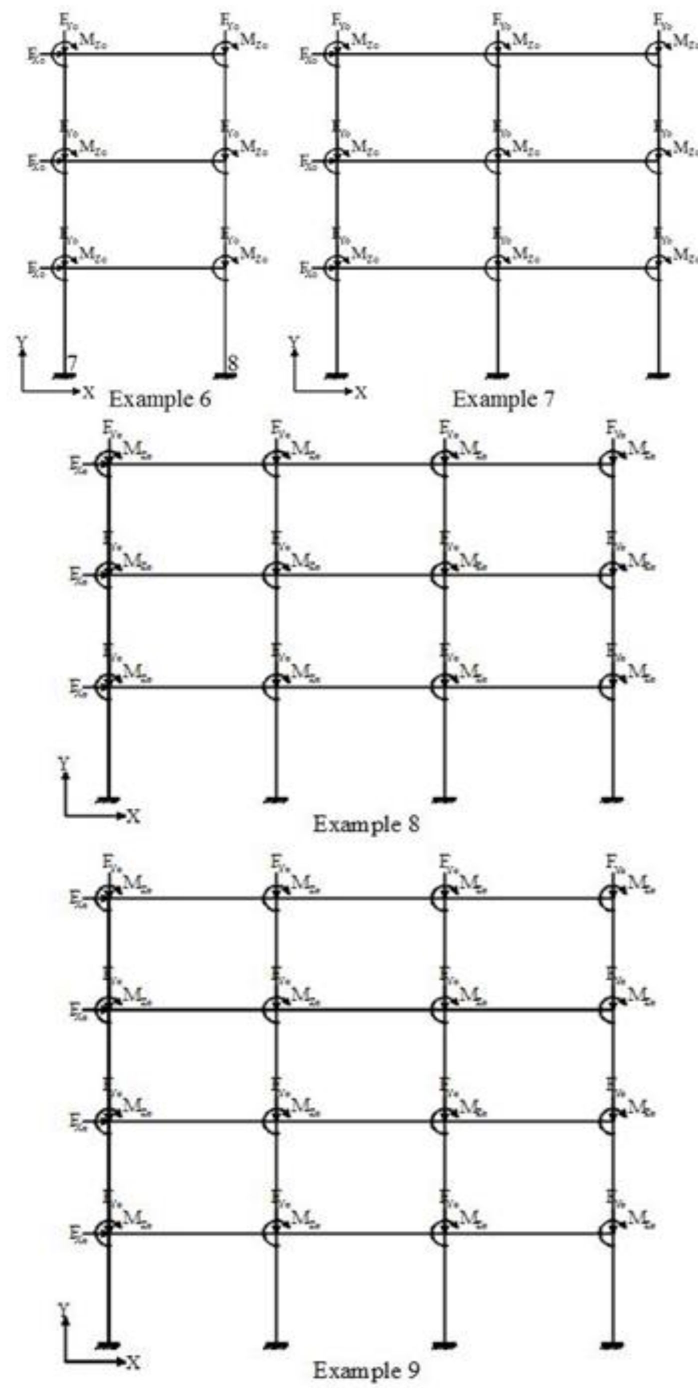

Fig. 6*. The configuration of the members of Examples 6 to 9*

It is clear that the MGS method significantly outperformed the GS method in all the nine Practical Examples (Fig. 8). Moreover, as the number of DOFs increases—especially by increasing the number of floors—the MGS becomes much more powerful than GS technique as MGS takes a considerably shorter analysis time. It can be noticed from Fig. 9 that the MGS* is remarkably faster than MGS, especially in case of a large number of DOFs as in Example 9.

***Table 1* Number** *of iterations of the 2D Practical Examples*

Practical Example	No. of DOFs	Number of iterations			
		Jacobi	GS	MGS	MGS*
Example 1	9	385	197	18	10
Example 2	12	422	208	16	9
Example 3	15	430	214	16	8
Example 4	18	---	700	72	36
Example 5	24	---	724	66	33
Example 6	18	---	1414	200	101
Example 7	27	---	1541	172	86
Example 8	36	---	1575	158	80
Example 9	48	---	2761	288	147

Table 2 *Analysis time of the 2D Practical Examples*

Practical Example	No. of DOFs	Analysis time (Sec)			
		Jacobi	GS	MGS	MGS*
Example 1	9	1.22	0.61	0.16	0.11
Example 2	12	1.66	0.86	0.15	0.11
Example 3	15	2.12	1.11	0.17	0.11
Example 4	18	---	4.20	0.6	0.39
Example 5	24	---	5.98	0.73	0.45
Example 6	18	---	8.46	1.61	1.05
Example 7	27	---	14.56	2	1.19
Example 8	36	---	20.95	2.45	1.44
Example 9	48	---	53.20	6.27	3.6

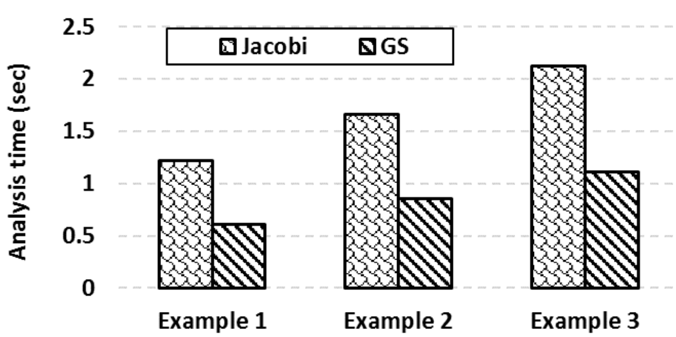

Fig. 7*. The analysis time of Jacobi and GS for Examples 1 to 3*

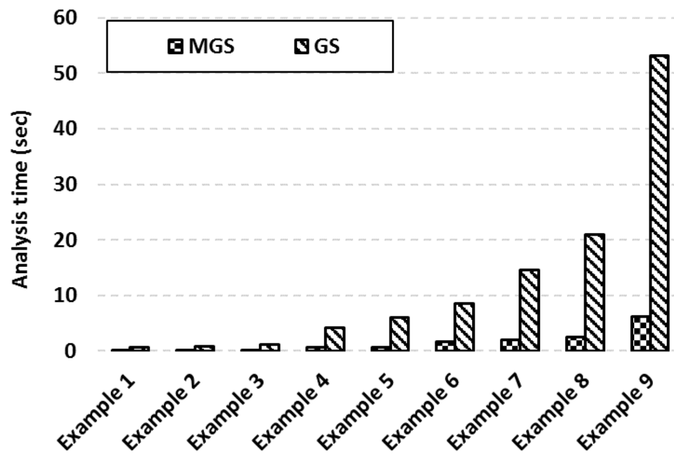

Fig. 8*. The analysis time of GS and MGS for Examples 1 to 9*

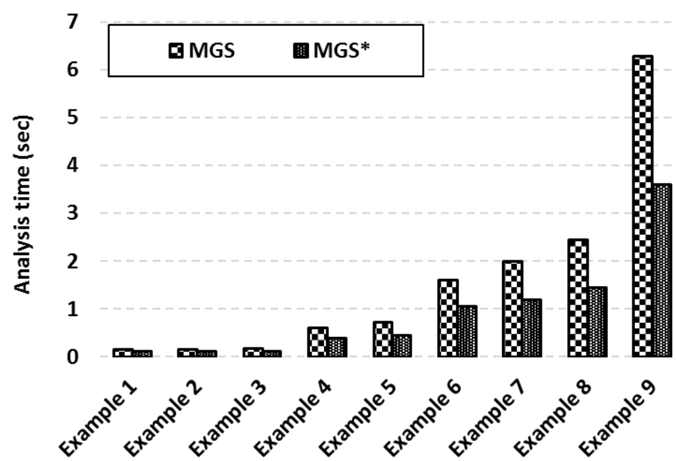

Fig. 9*. The analysis time of MGS and MGS* for Examples 1 to 9*

4 Conclusion

To recap, a new modified approach of Gauss-Seidel iterative technique has been introduced in this paper for solving the systems of linear equations resulting from applying the stiffness method for structural models using the finite element analysis. The algorithm of the new proposed "Modified Gauss-Seidel" (MGS) technique has been elaborated. Then, to assess the performance and efficiency of MGS compared to other Stationary techniques, some 2D Practical Examples have been solved using Jacobi, Gauss-Seidel (GS), and MGS. The results of those Examples led to the following conclusions:

- The jacobi iterative method is not comparable to the Gauss-Seidel (GS) method or the developed Modified Gauss-Seidel (MGS) method.
- The large number of floors has a worse impact on the solver's speed than the large number of floors. The more the number of floor increases, the slower both GS and MGS solvers become.
- Generally, the developed MGS technique is much more powerful than the traditional GS procedure.
- Furthermore, the idea of using the modified iteration cycle only that resulted in MGS* is advantageous as MGS* has been proved to be significantly faster than MGS especially

in case of a large number of unknowns.

- At a significantly large number of degrees of freedom (unknowns), especially due to increasing the number of floors, the GS becomes no more comparable to the developed MGS technique.

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