Characterization Of Strong And Weak Dominating χ- Color Number In A Graph

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Abstract: Strong dominating χ- color number of a graph G is defined as the maximum number of color classes which are strong dominating sets of G, and is denoted by $sd_\chi(G)$. Similarly, weak dominating χ- color number of a graph G is defined as the maximum number of color classes which are weak dominating sets of G, and is denoted by $wd_\chi(G)$. In both the cases, the maximum is taken over all χ-coloring of G. In this paper, some bounds for $sd_\chi(G)$ and $wd_\chi(G)$ are obtained and characterized the graphs for which strong dominating χ- color number and strong dominating χ- color number exist. Finally, Nordhaus-Gaddum inequalities for $sd_\chi(G)$ and $wd_\chi(G)$ is derived.

Index Terms: Dominating-χ-color number, Strong dominating χ-color number, Weak dominating χ-color number.

1 Introduction

In this paper, we consider finite, connected, undirected and simple graph $G = (V(G), E(G))$ with vertex set $V = V(G)$ and edge set $E = E(G)$). The number of vertices $|V(G)|$ of a graph G is called the order of G and the number of edges $|E(G)|$ of a graph G is called the size of G. The order and size is denoted by n and m respectively. In graph theory, coloring and dominating are two important areas which have been extensively studied. The fundamental parameter in the theory of graph coloring is the chromatic number $\chi(G)$ of a graph G which is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices receive the same color. If $\chi(G) = k$, we say that G is $k-$chromatic [1]. For any vertex $v\in V(G)$, the open neighborhood of v is the set $N(v) = \{u|\ uv \in E(G)\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. Similarly, for any set $S\subseteq V(G)$, $N(S) = \cup_{v\in S} N(v) - S$ and $N[S] = N(S)\cup S$. A set S is a dominating set if $N[S] = V(G)$. The minimum cardinality of a dominating set of G is denoted by $\gamma(G)$[4]. A set $D \subseteq V$ is a dominating set of G, if for every vertex $x\in V - D$ there is a vertex $y \in D$ with $xy\in E$ and D is said to be strong dominating set of G, if it satisfies the additional condition $\deg(x) \le \deg(y)$ [2]. The strong domination number $\gamma_{st}(G)$ is defined as the minimum cardinality of a strong dominating set. A set $S \subseteq V$ is called weak dominating set of G if for every vertex $u\in V - S$, there exists vertex $v\in S$ such that $uv\in E$ and $\deg(u) \ge \deg(v)$. The weak domination number $\gamma_w(G)$ is defined as the minimum cardinality of a weak dominating set and was introduced by Sampathkumar and PushpaLatha [3]. The number of maximum degree vertices of G is denoted by $n_\Delta(G)$ and the number of minimum degree vertices of G is denoted by $n_\delta(G)$.

2 Terminology

We start with more formal definition of dominating χ-color number of G [5]. Let G be a graph with $\chi(G) = k$. Let $C = V_1, V_2, ..., V_k$ be a k-coloring of G. Let d_C denotes the number of color classes in C which are dominating sets of G. Then $d_\chi(G) = \max_C d_C$ where the maximum is taken over all the k-colorings of G, is called the dominating χ-color number of G. Instead of dominating set in the definition of dominatingχ-color number, if we consider strong dominating set, then it is called strong dominatingχ-color number $sd_\chi(G)$ and, if we consider weak dominating set, then it is called weak dominating-χ-color number$wd_\chi(G)$[6]. Though substantial work has been carried out on domination and coloring parameters and related topics in graphs, there are only a few results concerning strong and weak domination in graphs. The new parameter, strong dominating -χ- color number and weak dominating $\chi -$ color number defined by us in [7]. The following are some perceived propositions,

Proposition 1 : For any graph$G$, $0 \le sd_\chi(G) \le d_\chi(G)$.

Proposition 2 : For any graph$G$, $0 \le wd_\chi(G) \le d_\chi(G)$.

Proposition 3 : For any cycleC_n; $sd_\chi(C_n) = \begin{cases} 3, & \text{if } n \equiv 3 (\text{mod} 6) \\ 2, & \text{otherwise} \end{cases}$

Proposition 4 : If W_n, is any wheel with n > 4, then $sd_\chi(W_n) = 1$.

3 Strong (Weak) Dominating χ- Color Number Equals to Zero

Arumugam et al. [8] observed that "Every graph contains a χ –coloring with the property that at least one color class is a dominating set in G." Strong dominating χ –color number and weak dominating χ –color number may not exist for all the graphs, even though every graph has dominating χ –color set and independent strong dominating set and weak dominating set. Because, in the definition of strong dominating χ –color number and weak dominating χ –color number, first priority goes to χ –coloring of a graph. To examine the necessary condition for $sd_\chi(G)$ and $wd_\chi(G)$ equals zero, we proved the following theorem.

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Theorem 1: If there exist a pair of non-adjacent vertices (u, v) such that for all $x \in N(u)$, $d(u) > d(x)$, and for all $y \in N(v)$, $d(v) > d(y)$, has distinct color in all possible χ –coloring of a graph G, then $sd_\chi(G) = 0$.

Proof:
Let D be a color class with vertex $u$of G. Let there exist a pair of non-adjacent vertices (u, v) such that for all $x \in N(u)$, $d(u) > d(x)$, and for all $y \in N(v)$, $d(v) > d(y)$, has distinct color in all possible χ –coloring of a graph G. Since the vertex u can not be strongly dominated by any vertices of G, the only one color class D may be a strong dominating χ –color set of G. But there is vertex $v \notin D$, that also cannot be strongly dominated by any other vertices. Therefore, $sd_\chi(G) = 0$.The following theorem is immediate for weak dominating χ –color number.

Theorem 2: If there exist a pair of non-adjacent vertices (u, v) such that for all $x \in N(u)$, $d(u) < d(x)$, and for all $y \in N(v)$, $d(v) < d(y)$, has distinct color in all possible χ – coloring of a graph G, then $wd_\chi(G) = 0$.

4 STRONG (WEAK) DOMINATING χ- COLOR NUMBER EQUALS TO ONE

Negation of theorem 6, gives the following theorem 3 that characterize the graphs for which $sd_\chi(G) \geq 1$and $wd_\chi(G) \geq 1$.

Theorem 3: If there is no pair of non-adjacent vertices (u,v) has distinct color in χ –coloring of a graph G,then $sd_\chi(G) \geq 1$ and $wd_\chi(G) \geq 1$.

Theorem 4: If there is no pair of non-adjacent vertices (u, v) such that for all $x \in N(u)$, $d(u) > d(x)$, and for all $y \in N(v)$, $d(v) > d(y)$, has distinct color in all possible χ –coloring of a graph G, then $sd_\chi(G) \geq 1$.

Theorem 5: If there is no pair of non-adjacent vertices (u, v) such that for all $x \in N(u)$, $d(u) < d(x)$, and for all $y \in N(v)$, $d(v) < d(y)$, has distinct color in all possible χ – coloring of a graph G, then $wd_\chi(G) \geq 1$.

Theorem 6: For any regular graph G, $sd_\chi(G) = wd_\chi(G) = d_\chi(G)$.

Proof : Since all the degree of the regular graph is same, all the dominating color classes are strong as well weak. Consequently, $sd_\chi(G) = wd_\chi(G) = d_\chi(G)$.

Corollary 1: For any complete graph K_n; $sd_\chi(K_n) = wd_\chi(K_n) = n$.

Theorem 7: For any bi-regular graph G,
(a) If $\Delta(G) \neq \delta(G)$, then $sd_\chi(G) = wd_\chi(G) = 1$.
(b) If $\Delta(G) = \delta(G)$, then $sd_\chi(G) = wd_\chi(G) = 2$.

Proof:
(a) Since $\Delta(G) \neq \delta(G)$, there are two partitions with different degree. If the maximum degree vertex partition and minimum degree vertex partition associated with color class C_1 and C_2 respectively, then C_1 is strong dominating color set and C_2 is weak dominating color set. Therefore, $sd_\chi(G) = wd_\chi(G) = 1$.
(b) Since $\Delta(G) = \delta(G)$, all the degree of the biregular graph is same, both the partitions are dominating color classes as well as strong and weak. Consequently, $sd_\chi(G) = wd_\chi(G) = 2$.

Corollary 2: For any star graph, $K_{1,n}$, $sd_\chi(K_{1,n}) = wd_\chi(K_{1,n}) = 1$.

Theorem 8: Let G be a path P_n, then
$$sd_\chi(P_n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } 3 \\ 2 & \text{otherwise} \end{cases}$$

Proof:
For $n = 1$or 3, it is clear that $sd_\chi(P_n) = 1$.
For $n = 2$, the path $P_2 = K_2$. $sd_\chi(P_n) = 2$.
Now our claim is $sd_\chi(P_n) = 2$, if $n > 3$. Since P_n is a bipartite graph, $d_\chi(P_n) = \chi(P_n) = 2$, let A and B be two dominating χ –color sets of P_n. Let u and v be the pendent vertices of P_n. If n is odd, then u,v ∈ A. In that case, all the vertices outside A can be strongly dominated by A-{u,v}. So, A strongly dominates B. All the vertices outside B can be strongly dominated by B, because the dominating set B has only vertices with maximum degree. If n is even, then u ∈ Aand v ∈ B. In that case, all the vertices outside A can be strongly dominated by A-{u}. So, A strongly dominates B. All the vertices outside B can be strongly dominated by B-{v}, So, B strongly dominates A. Consequently both A and B strong dominating χ –color sets of P_n. Thus, $sd_\chi(P_n) = 2$.

For $n = 2$, the path $P_2 = K_2$. $wd_\chi(P_n) = 2$. The next theorem gives the weak dominating χ –color sets of $P_n$for $n \neq 2$.

Theorem 9: Let G be a path $P_n, n \neq 2$, then
$$wd_\chi(P_n) = \begin{cases} 1 & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$$

Proof:
Since P_n is a bipartite graph, $d_\chi(P_n) = \chi(P_n) = 2$, let A and B be two dominating χ –color sets of P_n. Let u and v be the pendent vertices of P_n. If n is odd, then u,v ∈ A. In that case, all the vertices outside A can be weakly dominated by A-{u,v}. So, A weakly dominates B. But the pendent vertices u,v are outside B, they cannot be weakly dominated by any vertex of B. Even though B is a dominating χ –color set, it is not a weak dominating χ –color set. Consequently, $wd_\chi(P_n) = 1$. If n is even, then u ∈ Aand v ∈ B. In that case, both A and B are not weak dominating χ –color sets. Because, the pendent vertices u and v is only dominated by its support vertices , which has degree two. Thus $wd_\chi(P_n) = 0$.

Theorem 10 : For any graphs G and H,
$$sd_\chi(G \cup H) = \min\{sd_\chi(G), sd_\chi(H)\}.$$
Proof:
Let $D_1, D_2, \ldots, D_{sd_\chi(H)}$ be the strong dominating χ –color sets of H and let $C_1, C_2, \ldots, C_{sd_\chi(G)}$ be the strong dominating χ –color sets of G. Without loss of generality, let us assume that $sd_\chi(G) > sd_\chi(H)$.
Now combine the vertices of $D_i$and C_i, for all $1 \leq i \leq sd_\chi(H)$.Then
$D_1 \cup C_1, D_2 \cup C_2, \ldots, D_{sd_\chi(H)} \cup C_{sd_\chi(H)}, C_{sd_\chi(H)+1}, \ldots, C_{sd_\chi(G)}$ are the

color classes of $G \cup H$. By definition of $G \cup H$, no vertices of G dominates the vertices of H and vice versa. Therefore, only $D_1 \cup C_1, D_2 \cup C_2, \ldots, D_{sd_\chi(H)} \cup C_{sd_\chi(H)}$ are the strong dominating χ −color sets of $G \cup H$. Hence $sd_\chi(G \cup H) = \min\{sd_\chi(G), sd_\chi(H)\}$.

5 Existence of graphs with $sd_\chi(G) = 1$ and $wd_\chi(G) = 1$

Arumugam et al. [8] showed that "For every integer $k \geq 0$, there exists a connected graph G with $\delta(G) = k$ and $d_\chi(G) = 1$." In the proof, they begin with the statement "For $k = 0$, take $G = \overline{K}_n$." But $\overline{K}_n$ is not connected graph for $n > 1$, which contradicts the statement "there exists a connected graph". So, we need to start the proof as: For $k = 0$, take $G = K_1$. Similarly, we next show that even if the minimum degree of G is arbitrarily large, the strong dominating χ −color and weak dominating χ −color number number may be 1.

Theorem 11 : For every integer $k \geq 0$, there exists a graph G with $\delta(G) = k$ and $sd_\chi(G) = 1$ and $wd_\chi(G) = 1$.

Proof :

For $k = 0$, take $G = K_1$. Hence we may assume that $k > 0$. Let G_k be obtained from a complete bipartite graph with partite set V_1 and V_2 each of cardinality k by adding a new vertex v and adding all the edges between vertex v and vertices of V_1. By construction, $\delta(G_k) = k$. Since G_k is a bipartite graph and $|V_1| \neq |V_2|$, $sd_\chi(G) = 1$ and $wd_\chi(G) = 1$

Theorem 12 : For all integers $a \geq b \geq 0$, there exists a graph G with $\chi(G) = a$ and $sd_\chi(G) = b$.

Proof:

For $a = b$, take $G = K_a$. Hence we may assume that $a > b$.
Suppose that $a > 1$ and $b = 1$. Let G be obtained from a complete graph K_a by adding a new vertex v and adding an edge between vertex v and any vertices of K_a. By construction, $\chi(G) = a$ and $sd_\chi(G) = 1$. Thus $\chi(G) = a$ and $sd_\chi(G) = b$.
Suppose that $a > 1$ and $b > 1$. Let G be obtained from complete graphs K_a with vertex set $\{v_1, v_2, \ldots, v_a\}$ and K_b with vertex set $\{u_1, u_2, \ldots, u_b\}$ by adding b non-adjacent edges $\{v_1u_1, v_2u_2, \ldots, v_bu_b\}$ between them.
If G is colored by the coloring function $c(v_i) = i$ and $c(u_j) = j + 1$, then $\chi(G) = a$. Since the coloring of G has b color classes that are strong dominating sets in G, say the color classes associated with the colors $2, \ldots, b + 1$, we have $sd_\chi(G) \geq b$.
The color classes $\{u_1, v_2\}, \{u_2, v_3\}, \ldots, \{u_b, v_{b+1}\}$ are strong dominating sets of G. The remaining $(a - b)$ color classes $\{v_1\}, \{v_{b+2}\}, \{v_{b+3}\}, \ldots, \{v_a\}$ are not dominating sets of G. Consequently, $\chi(G) = a$ and $sd_\chi(G) = b$. Correspondingly, we can show the following theorem for weak dominating χ −color number.

Theorem 13: For all integers $a \geq b \geq 0$, there exists a graph G with $\chi(G) = a$ and $wd_\chi(G) = b$.

6 Nordhaus-Gaddum Inequalities

For any graph G with parameter ψ, sharp upper and lower bounds for both $\psi(G) + \psi(\overline{G})$ and $\psi(G) \cdot \psi(\overline{G})$ are referred as Nordhaus-Gaddum inequalities. In this section, some bounds and these inequalities are derived for $sd_\chi(G)$ and $wd_\chi(G)$.

Observation 1: For all graphs G,
$sd_\chi(G) \leq \delta + 1$ and $wd_\chi(G) \leq \delta + 1$.

Theorem 14: If any graph G is a non-regular graph with n vertices and it has a connected complement, denoted by $\overline{G}$, then

(a) $sd_\chi(G) \leq n_\Delta(G)$ and $wd_\chi(\overline{G}) \leq n_\Delta(G)$.
(b) $sd_\chi(\overline{G}) \leq n_\delta(G)$ and $wd_\chi(G) \leq n_\delta(G)$.
(c) $sd_\chi(G) + sd_\chi(\overline{G}) \leq n$.
(d) $wd_\chi(G) + wd_\chi(\overline{G}) \leq n$.
(e) $sd_\chi(G) + wd_\chi(G) \leq n$.
(f) $sd_\chi(\overline{G}) + wd_\chi(\overline{G}) \leq n$.

Proof:

It is necessary to have a maximum degree and minimum degree vertices of G in strong dominating χ − color set of G respectively. And number of maximum degree $n_\Delta(G)$ and minimum degree $n_\delta(G)$ vertices of G equal to the number of minimum degree and maximum degree vertices of $\overline{G}$ respectively. Due to this fact, (a) and (b) can be easily proved. From (a) and (b), $sd_\chi(G) + sd_\chi(\overline{G}) \leq n_\Delta(G) + n_\delta(G)$. Since G is not a regular graph, $sd_\chi(G) < n$, $wd_\chi(G) < n$ and $n_\Delta(G) + n_\delta(G) = n$. The following are immediate from (a) and (b).

(c) $sd_\chi(G) + sd_\chi(\overline{G}) = n_\Delta(G) + n_\delta(G) \leq n$.
(d) $wd_\chi(G) + wd_\chi(\overline{G}) = n_\Delta(G) + n_\delta(G) \leq n$.
(e) $sd_\chi(G) + wd_\chi(G) = n_\Delta(G) + n_\delta(G) \leq n$.
(f) $sd_\chi(\overline{G}) + wd_\chi(\overline{G}) = n_\delta(G) + n_\Delta(G) \leq n$.

Further, we can sharpen the bounds of $sd_\chi(G)$ and $wd_\chi(G)$ in the succeeding theorems. Let V_Δ and V_δ be the set of all vertices which has maximum and minimum degree of a graph G respectively.

Theorem 15: Let G be any graph, then $sd_\chi(G) \leq \chi(\langle V_\Delta \rangle)$ and $wd_\chi(G) \leq \chi(\langle V_\delta \rangle)$.

Proof:

Let G be any graph. The maximum degree vertices of G occur minimum $\chi(\langle V_\Delta \rangle)$ number of color classes of G. If all the maximum degree vertices occur in $\chi(\langle V_\Delta \rangle)$ number of color classes, then $sd_\chi(G) \leq \chi(\langle V_\Delta \rangle)$. If not, there are maximum $n_\Delta(G) - \chi(\langle V_\Delta \rangle)$ vertices occur in $\chi(G) - \chi(\langle V_\Delta \rangle)$ number of color classes which are not strong dominating sets of G. Because such maximum degree vertices are non adjacent and has different color with $\chi(\langle V_\Delta \rangle)$ number of maximum degree vertices. Also, those vertices cannot be strongly dominated by any other vertices of G. In that case, $sd_\chi(G) = 0$. Consequently, $sd_\chi(G) \leq \chi(\langle V_\Delta \rangle)$. Similarly, we can prove that $wd_\chi(G) \leq \chi(\langle V_\delta \rangle)$.

Theorem 16: Let G be any graph, then $sd_\chi(G) \leq \omega(\langle V_\Delta \rangle)$ and $wd_\chi(G) \leq \omega(\langle V_\delta \rangle)$.

Proof:

Let G be any graph. The maximum degree vertices of G occur minimum $\chi(\langle V_\Delta \rangle)$ number of color classes of G. It is clear that, out of this $\chi(\langle V_\Delta \rangle)$ number of color classes of G, $\omega(\langle V_\Delta \rangle)$

number of color classes of G can be a strong dominating set of G. Even though the remaining color classes of G has maximum degree vertices, they not dominate atleast one maximum degree vertex of G. So, they are not strong dominating sets of G. Therefore, $sd_\chi(G) \leq \omega(\langle V_\Delta \rangle)$. Similarly we can prove that $wd_\chi(G) \leq \omega(\langle V_\Delta \rangle)$.

Corollary 3: If $\langle V_\Delta \rangle$ is a null graph, then $sd_\chi(G) \leq 1$

Corollary 4: If $\langle V_\delta \rangle$ is a null graph, then $wd_\chi(G) \leq 1$.

Theorem 17: Let G be any graph with n vertices. Then

(a) $0 \leq sd_\chi(G) + sd_\chi(\overline{G}) \leq n_\Delta + n_\delta$ and $0 \leq sd_\chi(G) \cdot sd_\chi(\overline{G}) \leq n_\Delta \cdot n_\delta$

(b) $0 \leq wd_\chi(G) + wd_\chi(\overline{G}) \leq n_\Delta + n_\delta$ and $0 \leq wd_\chi(G) \cdot wd_\chi(\overline{G}) \leq n_\Delta \cdot n_\delta$

Proof: Since $0 \leq sd_\chi(G) \leq n_\Delta$ and $0 \leq sd_\chi(\overline{G}) \leq n_\delta$, we can easily derive(a) and (b).

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