Construction Of Lyapunov Functions For Some Fourth Order Nonlinear Ordinary Differential Equations By Method Of Integration

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Abstract: In giving adequate attention to some qualitative properties of solutions in ordinary differential equations, Lyapunov functions is quite indispensable. The process of tackling some problems in the application is the construction of appropriate Lyapunov functions for some nonlinear fourth order differential equation. Our focus is finding V(x), a quadratic form and positive definite also finding U(x) which is positive definite such that the derivative of V with respect to time would be equal to the negative value of U(x) In this work, we adopted the pre-multiplication of the given differential equation by $\dddot{X}$ and thereafter we integrated with respect to t form $t=0$ to $t=T$. We obtained a Lyapunov function candidate for a fourth order differential equation or its scalar equation.

Index Terms: Lyapunov Functions, Linear systems, non linear systems Negative definite, Positive definite, Linear and non linear systems

1. Introduction

Lyapunov functions are useful tools in determining stability, asymptotic stability, uniform stability, global stability or outright instability of differential system and boundedness of solution of a real scalar fourth-order differential equation (Ogundare 2012, Ezeilo et al 2010, Omeike, 2008, Afuwape, 2010) Asymptotic stability is intimately linked to the existence of a 'Lyapunov' function, that is, a proper, non-negative function varnishing only on an invariant set and decreasing along those trajectories of the system not evolving in the invariant set. Lyapunov proved that the existence of a Lyapunov function guarantees asymptotic stability and for linear time-invariant systems also showed the converse statement that asymptotic stability implies the existence of a Lyapunov function (Lars et al, 2006).

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Stability is a very important problem in the theory and application of differential equations, and an effective method for studying the stability of differential equation is the second method of Lyapunov [Omeike, 2008] Lyapunov functions originated from a Russian mathematician Aleksandra M. Lyapunov. The basis for all what he was proving is entirely on the well known fact that towards the equilibrium position, the total energy in a system is either constant or decreasing. Lyapunov functions have been constructed for linear equations on the platform that given any quadratic positive definite V, there exists another positive definite function U such that $\dot{V}=-U$ and for the nonlinear case, a correlation is taken between the constant coefficient equations of linear and nonlinear equations which leads to the appropriate Lyapunov functions for the nonlinear case [Ezeilo et al, 2010] Problems and complex computations are encountered in trying to construct appropriate Lyapunov function candidates for nonlinear fourth order differential systems. However, our method for construction of Lyapunov functions for nonlinear fourth order differential lies on the fact that given any real system

$$\dot{x}=Ax \quad . \quad . \quad . \qquad (1.0.0)$$

$x \in \mathbb{R}^n$, where A is a constant matrix and that A has all its eigen values with negative real parts. Then it is known from the general theory that corresponding to any positive definite quadratic form $U(x)$ there exists another positive definite quadratic form $V(x)$ such that

$$\dot{V} = -U \quad . \quad . \quad . \qquad (1.0.1)$$

along the solution paths of (1.0.0). This result in (1.0.1) has been extended to hold for positive semi definite quadratic $U(x)$ as well. As a matter of fact, our basis for construction of Lyapunov function in this work would ultimately satisfy equation (1.0.1) We consider V to be positive definite function, then if V always decreases, then it must reach zero eventually. That is, for a stable system, all trajectories must move so that the values of V are decreasing. This is similar to the energy argument for of mechanical systems. To relate V to the system dynamics, we compute $\dot{V}$

$$\dot{V} = \tfrac{\partial V}{\partial t} + \sum_i \tfrac{\partial V}{\partial x_i} \dot{x}_i$$

$$= \tfrac{\partial V}{\partial t} + \nabla V^T f(x)$$

where

$$f(x) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}$$

and

$$\nabla V^T = \left[\tfrac{\partial V}{\partial X_1}, \tfrac{\partial V}{\partial X_2}, \tfrac{\partial V}{\partial X_3}, \tfrac{\partial V}{\partial X_4} \right]$$

We need $\dot{V}$ to be negative definite for our expectation to be true. Also, if we consider the autonomous linear system (1.0.0), and we want to investigate the stability of the system using Lyapunov theory, for the sake of argument, let

$$V = x^T P x$$

where P is a positive definite matrix. Then,

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x}$$

$$= x^T \left\{ A^T P + PA \right\} x$$

Since $\dot{X} = AX$ and $\dot{X}^T = (AX)^T = x^T A^T$

Let $A^T P + PA = -Q$, where Q is positive definite. If Q is positive definite then the linear system is globally asymptotically stable. $A^T P + PA = -Q$ is called the Lyapunov equation. So, in solving for P, we need to first of all accept that A is stable, so that any $Q > 0$ will yield a $P > 0$, the usual approach though is to set $Q = 1$, then solve for P (Hedrick and Girard, 2005). In Lyapunov's second method, the theorem provides a sufficient condition, not a necessary condition. Consider the system

$$\dot{x} = f(x,t),\ f(0,t) = 0 \quad \forall\ t$$

If a scalar function is defined such that

(i) $V(0,t) = 0$

(ii) $V(x,t)$ is positive, that is, there exists a continuous non decreasing scalar function $\alpha(0) = 0$ and $\forall\ x \neq 0, 0 < \alpha(\|x\|) < V(x,t)$

(iii) $\dot{V}(x,t)$ is negative definite, that is $\dot{V}(x,t) \leq -\Upsilon(\|x\|) < 0$ where Υ is a continuous non-decreasing scalar function such that $\Upsilon(0) = 0$

(iv) $V \leq \beta(\|x\|)$ where β is a continuous non-decreasing function and $\beta(0) = 0$ that is V is decrescent, the Lyapunov function is upper bounded.

(v) V is radically unbounded, that is, $\alpha(\|x\|) \to \infty$ as $\|x\| \to \infty$

We are only trying to bring to mind that Asymptotic stability requires $\dot{V}$ to be negative definite, stability requires $\dot{V}$ to be negative semi definite, condition (iv) yields uniformity for the time-varying system and Global stability is given by condition (v). Ordinarily, no matter where one starts, equilibrium is returned we refer to it as uniform. For asymptotic stability, we say that no matter how large the perturbation, the system returns back to origin; and in the large (globally), it is true everywhere (Ademola et al,2011) A more general statement for globally asymptotically stable autonomous system (1.0.2)with f continuous and let V be a scalar with continuous first partial derivatives. Assume that

(i) for some $K > 0$, the region Ω_k defined by $V(x) < K$ is bounded

(ii) $\dot{V} \le 0$ for all x in Ω_k

let R be the set of all points within Ω_i where $\dot{V} = 0$. Let M be the largest invariant set in R. then every solution $x(t)$ originating in Ω_k tends to M as t tends to infinity. Since Lyapunov proposed his famous theory on stability of motion, numerous methods have been proposed for deriving suitable Lyapunov functions to study the stability and boundedness of solutions of certain second-third-forth-fifth and sixth order non-linear differential equations; see for example [Anderson 1968, Barbasin 1968] The other Lyapunov's theorem considered $V(x)$ such that $V(0) = 0$ and

$$V(x) > 0 \quad \forall x \in D/\{0\} \quad \dot{V}(x) \le 0, \quad \forall x \in D$$

then the origin is stable, moreover if

$$\dot{V}(x) <, \ \forall x \in D/\{0\}$$

then the origin is asymptotically stable. Furthermore, if

$$V(x) > 0, \ \forall x \ne 0$$
$$\|x\| \to \infty \Rightarrow V(x) \to \infty$$

and $\dot{V}(x) < 0, \ \forall x \ne 0$, then the origin is globally asymptotically stable. So we can observe that the origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative definite, it is globally asymptotically stable if the conditions for asymptotic stability holds globally and $V(x)$ is radially unbounded. A continuously differentiable function $V(x)$ satisfying the conditions for stability is called a Lyapunov function. The surface $V(x) = c$ for some $c > 0$, is called a Lyapunov surface or a level surface. A geometric integrator for a system of ordinary differential equations with a Lyapunov function V should preserve V as a Lyapunov function for the discrete system (Grimm and Quispel, 2005) (Bainov, 1997) by means of piecewise continuous auxiliary functions which are analogues of the classical Lyapunov's functions, obtained sufficient conditions for the existence of periodic solutions of a linear system of impulsive differential difference equations. The impulse takes place at fixed moments. Their investigations were carried out by using minimal subsets of a suitable space of piecewise continuous functions, by the elements of which the derivatives of the piecewise continuous auxiliary functions are estimated. The Lyapunov stability of periodic solutions were established from the generalized higher-order neutral functional differential equation.

$$\left(\varphi_p\left(x(t) - Cx(t-\sigma)\right)^{l}\right)^{n-1} = F\left(t, x(t), x'(t), \ldots, x^{(l-1)}(t)\right)$$

into a system of first-order differential equations and then applying Mawhin's continuation theory and some new inequalities, with this sufficient conditions for the existence of periodic solution for the equation was established (Jingli etal, 2011). Mohammed etal (2013) in their lecture on dynamic systems and control opined that a continuous-time system

$$\dot{x}(t) = Ax(t)$$

is asymptotically stable if and only if all the eigenvalues of A are in open left half plane, and also showed that this result can be inferred from Lyapunov theory. A consequence of this is that stability can be assessed by methods that may be computationally simpler than Eigen analysis. So, quadratic Lyapunov functions and the associated mathematics turn up are worth mastering in the context of stability evaluation. In this paper, we address the issue of construction of Lyapunov function for the linear systems and then we present and verify Lyapunov indirect methods for proving stability of a non-linear system with interest in fourth-order.

2. Statement of Problems. Preliminaries, Definitions.

Consider the fourth-order differential equation

$$\ddot{\ddot{x}} + a\dddot{x} + b\ddot{x} + c\dot{x} + dx = 0 \quad . \quad . \quad . \qquad (2.0.1)$$

where a, b, c and d are constants with

$$a > 0,\ b > 0,\ c > 0 \text{ and } d > 0 \quad . \quad . \quad . \qquad (2.0.2)$$

The equation (2.0.1) is equivalent to the following four systems.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= w \\ \dot{w} &= -aw - bz - cy - dx \end{aligned} \quad . \quad . \quad . \qquad (2.0.3)$$

The system (2.0.3) have negative real parts if and only if $a > 0,\ b > 0,\ c > 0$ and $d > 0$. There is therefore need to have a positive definite continuous quadratic function V and another positive quadratic form U such that

$$\dot{V} = -U \quad . \quad . \quad . \qquad (2.0.4)$$

along the solution paths of (2.0.1) or (2.0.3). Before now, the result in equation (2.0.4) has been extended and is established to hold for positive semi definite quadratic $U(x)$ as well. It is our interest therefore to construct a Lyapunov function that would ultimately satisfy equation (2.0.4)

2.1 Definitions

(i) A continuous function

$$V(x,t) = V(x_1, x_2, ..., x_n, t)$$

is positive definite if

$$\lim_{\|x\|\to 0} V(x,t) = 0$$

and there exists

$$\phi(\|x\|)$$

such that

$$V(x,t) > \phi(\|x\|) \quad . \quad . \quad . \qquad (2.0.5)$$

(ii) A continuous function

$$V(x,t) = V(x_1, x_2, ..., x_n, t)$$

is positive semi definite if

$$\lim_{\|x\|\to 0} V(x,t) = 0$$

and there exist

$$\phi(\|x\|)$$

such that

$$V(x,t) \ge \phi(\|x\|) \quad . \quad . \quad . \qquad (2.0.6)$$

(iii) A continuous function

$$V(x,t) = V(x_1, x_2, ..., x_n, t)$$

is negative definite if

$$\lim_{\|x\|\to 0} V(x,t) = 0$$

and there exists

$$\phi(\|x\|)$$

such that

$$V(x,t) < -\phi(\|x\|) \quad . \quad . \quad . \qquad (2.0.7)$$

(iv) A continuous function

$$V(x,t) = V(x_1, x_2, ..., x_n, t)$$

is negative semi definite if

$$\lim_{\|x\|\to 0} V(x,t) = 0$$

and there exists

$$\phi(\|x\|)$$

such that

$$V(x,t) \le -\phi(\|x\|) \quad . \quad . \quad . \qquad (2.0.8)$$

(v) A continuous function

$$V(x,t) = V(x_1, x_2, ..., x_n, t)$$

is indefinite if it assumes both positive and negative values in any arbitrary neighbourhood of the origin in a Domain D.

at a glance we have

$V(0)=0, V(x)\geq 0$ for $x\neq 0$ (Positive semi definite)

$V(0)=0, V(x)>0$ for $x\neq 0$ (Positive definite)

$V(0)=0, V(x)\leq 0$ for $x\neq 0$ (Negative semi definite)

$V(0)=0, V(x)<0$ for $x\neq 0$ (Negative definite)

$\|x\|\to\infty, V(x)\to\infty$ (Radially undounded)

In another Lyapunov theorem, we reaffirm the definitions in this context. Given a differential equation

$$\dot{x}=f(t,x),\ f(t,o)=0 \quad . \quad . \quad . \qquad (2.0.9)$$

f is continuous in (t,x)

(vi) Suppose there exists a C' function $V:R^n\to R$ with the following properties

(a) V is positive definite

(b) The time derivative of V, V along the solution path of (2.0.9) is negative semi definite. Then the trivial solution $x\equiv 0$ is stable in the sense of Lyapunov.

(vii) Suppose there exists a C' function

$V:R^n\to R$ with the following properties

(a) V is positive definite

(b) The time derivative of V, $\dot{V}$ along the solution path of equation (2.0.9) is negative definite. Then the trivial solution $x\equiv 0$ is asymptotically stable in the sense of Lyapunov.

(viii) Suppose there exists a C' function

$V:R^n\to R$ with the following properties

(a) V is positive definite

(b) The time derivative of V, $\dot{V}$ along the solution path of equation (2.0.9) is positive definite and in every neighbourhood of the origin there exists x_o such that $V(x_0)>0$, then the trivial solution $x\equiv 0$ of equation (2.0.9) is unstable in the sense of Lyapunov.

2.2 Fundamental Theorem

Given the system (2.0.9) and suppose there exists a C' functional

$V:R^n\to R$ with the following properties

(a) V is positive definite

(b) The time derivative of V, $\dot{V}$ along the solution path of equation (2.0.9) is negative definite.

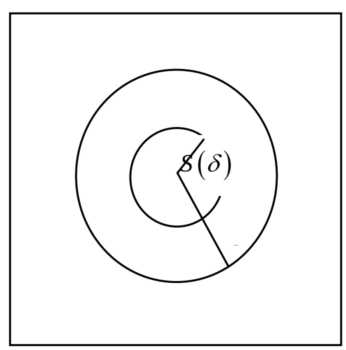
(c) Then the trivial solution $x\equiv 0$ of equation (2.0.9) is asymptotically stable.

Proof:

Let $S(\xi)$ be a sphere in R^2

$$S(\xi)=\{x\in R^2 : \|x\|=\xi\}$$

$$f: D\to R^2;\ D=\{x \mid \|x\|\leq \ \}$$



There exists $\delta>0$ such that V is a non increasing function along all the solution of $x(t,x_0)$ with $\|x_0\|<\delta$ Since V is positive definite in D and it is continuous, $V(x(t,x_0))$ approaches a limit $V(0)$ as $t\to\infty$. We assert that , then since $V(0)=\infty\ V(x)>0$, $x\neq 0$, then we have proved that $X(t,x_0)\to 0$ as $t\to\infty$.

Suppose not, that is $V(0) > 0$, then there exists $\xi > 0$ such that $V(x) < V(0)$ if $\|x\| < \xi$ (from continuity). We assert that the trajectory $x(t, x_0)$ will never enter $S(\xi) : \|x\| < \in$, where ξ is the radius of the sphere $S(\xi)$ From the hypothesis $-\dot{V}$ is positive definite, therefore $\frac{d}{dt}V(x,t) \le -M$ and on integrating we have,

$$V(x,t) - V(x,0) < \int_0^t -Mdt$$

Therefore,

$$V(x(t)) \le V(x_0) - mt$$

If we let $t \to \infty$, then

$$V(x(t)) \le V(x_0) - mt \to \infty$$

Hence a contradiction to the assumption that $V(0) > 0$ is impossible

3. Methodology and Discussions

The system (2.1) under investigation is

$$\ddddot{x} + a\dddot{x} + b\ddot{x} + c\dot{x} + dx = 0$$

resulting to the four scalar first order shown as follows:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= w \\ \dot{w} &= -aw - bz - cy - dx \end{aligned}$$

leading to the matrix

$$\dot{X} = AX = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -d & -c & -b & -a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \ldots \qquad (3.0.1)$$

where a, b, c, d > 0 for the system to have a negative real path. Consider,

$$2V = K_1x^2 + K_2y^2 + K_3z^2 + K_4w^2 + 2K_5xy + 2K_6xz + 2K_7xw + 2K_8yz + 2K_9yw + 2K_{10}zw \quad \cdots$$
(3.0.2)

There is a need to obtain the derivative with respect to t

$$\begin{aligned} 2\dot{V} = {} & 2K_1x\dot{x} + 2K_2y\dot{y} + 2K_3z\dot{z} + 2K_4w\dot{w} + 2K_5(y\dot{x} + x\dot{y}) + 2K_6(z\dot{x} + x\dot{z}) \\ & + 2K_7(w\dot{x} + x\dot{w}) + 2K_8(z\dot{y} + y\dot{z}) + 2K_9(w\dot{y} + y\dot{w}) + 2K_{10}(w\dot{z} + z\dot{w}) \end{aligned} \ldots (3.0.3)$$

$$\begin{aligned} \dot{V} = {} & K_1x\dot{x} + K_2y\dot{y} + K_3z\dot{z} + K_4w\dot{w} + K_5(y\dot{x} + x\dot{y}) + K_6(z\dot{x} + x\dot{z}) + K_7(w\dot{x} + x\dot{w}) \\ & + K_8(z\dot{y} + y\dot{z}) + K_9(w\dot{y} + y\dot{w}) + K_{10}(w\dot{z} + z\dot{w}) \end{aligned} \ldots (3.0.4)$$

But from equation (2.0.3)

$$\begin{aligned} \dot{V} = {} & K_1xy + K_2yz + K_3zw + K_4w(-aw - bz - cy - dx) + K_5(yy + xz) + K_6(zy + xw) + K_7(wy + x(-aw - bz - cy - dx)) \\ & + K_8(zz + yw) + K_9(wz + y(-aw - bz - cy - dx)) + K_{10}(ww + z(-aw - bz - cy - dx)) \end{aligned}$$

bringing the terms and their respective coefficients

Terms	Coefficients
x^2	$-dk_7$
y^2	$k_5 - ck_9$
z^2	$k_8 - bk_{10}$
w^2	$k_{10} - ak_4$
xy	$k_1 - ck_7 - dk_9$
xz	$k_5 - bk_7 - dk_{10}$
xw	$k_6 - dk_4 - ak_7$
yz	$k_2 + k_6 + bk_9 - ck_{10}$
yw	$k_7 - ck_4 + k_8 - ak_9$
zw	$k_3 - bk_4 + k_9 - ak_{10}$

. . .3.0.6

Next is to determine $\dot{V}$ such that one of the following conditions hold:

(i) $\dot{V} \le -\alpha x^2$
(ii) $\dot{V} \le -\alpha y^2$
(iii) $\dot{V} \le -\alpha z^2$
(iv) $\dot{V} \le -\alpha w^2$
(v) $\dot{V} \le -\alpha\left(x^2 + y^2 + z^2 + w^2\right)$

. . . 3.0.7

For the realization of any of these cases, one is required to impose conditions by realization of any of these terms as follows:

$k_7 = 0$ . . . (3.0.8)a

$ak_4 - k_{10} = 0$ . . . (3.0.8)b

$k_8 - bk_{10} = 0$ . . . (3.0.8)c

$k_5 - ck_9 > 0$ . . . (3.0.8d)

$k_{10} = ak_4$ from equation (3.0.8b)

$k_8 = bk_{10} \Rightarrow k_8 = abk_4$

Since $k_7 = 0$ and from equation (3.0.6)

$k_1 = dk_9$ . . . (3.0.9a)

$k_5 = dk_{10}$ . . . (3.0.9b)

$k_6 = dk_4$ . . . (3.0.9c)

$k_2 = ck_{10} - k_6 - b$ . . . (3.0.9d)

$k_8 = ck_4 + ak_9$ . . . (3.0.9e)

$k_3 = ak_{10} + bk_4 - k_9$ . . . (3.0.9f)

Recall that one interest is on equation (3.0.8d)

$$k_5 - ck_9 > 0$$

In (3.0.9b), $k_5 = dk_{10}$, thus

$dk_{10} - ck_9 > 0$ . . . (3.1.0)

In (3.0.9f), $k_9 = ak_{10} + bk_4 - k_3$ or In (3.0.9e) $k_8 = ck_4 + ak_9$

But

$k_8 = abk_4$, thus

$abk_4 = ck_4 + ak_9$

$\therefore \quad k_9 = bk_4 - \frac{c}{a}k_4$

From (3.1.0) and (3.0.8b)

$$adk_4 - c\left\{bk_4 - \frac{c}{a}k_4\right\} > 0$$

$$a^2dk_4 - acbk_4 + c^2k_4 > 0$$

$$\left(a^2d + c^2 - abc\right)k_4 > 0$$

By setting $k_4 = 1$ and substituting into equation (3.0.2) yields positive definite function V as one substitute the following

$k_1 = dbk_4 - \frac{dc}{a}k_4$

$k_2 = ack_4 + \frac{bc}{a}k_4 - dk_4 - b^2k_4$

$k_3 = a^2k_4 + \frac{c}{a}k_4$

. . . 3.0.7

$k_6 = dk_4$

$k_7 = 0$

$k_8 = abk_4$

$k_9 = bk_4 - \frac{c}{a}k_4$

$k_{10} = ak_4$

Then ploughing them back into equation (3.0.2) gives

$$2V = dK_4\left(b - \frac{c}{a}\right)x^2 + K_4\left(ac + \frac{bc}{a} - d - b^2\right)y^2 + K_4\left(a^2 + \frac{c}{a}\right)z^2 + w^2 + 2K_4adxy + 2K_4dxz + 2K_4abyz + 2K_4\left(b - \frac{c}{a}\right)yw + 2K_4awz \quad \text{. . . ..(3.1.1)}$$

By setting $k_4 = 1$ and dividing both sides by 2, we have

$$V = \frac{d}{2}\left(b - \frac{c}{a}\right)x^2 + \frac{1}{2}\left(ac + \frac{bc}{a} - d - b^2\right)y^2 + \frac{1}{2}\left(a^2 + \frac{c}{a}\right)z^2 + \frac{1}{2}w^2 + adxy + dxz + abyz + \left(b - \frac{c}{a}\right)yw + awz$$

This is positive definite provided that $ab > c$ and if we so choose that $a = b = c = d > 1$ This is an indication of positive definiteness and the corresponding time derivative

$$\dot{V} = -\left(a^2d + c^2 - abc\right)y^2 \quad . \; . \; . \qquad (3.1.3)$$

since equation (4.1.3) satisfies V defined by equation (3.1.2) satisfied equation (1.0.1) if U is replaced by $\left(a^2d + c^2 - abc\right)y^2$, then V defined by (3.1.2) is a Lyapunov function for the fourth order system (2.0.1)

3.1 Lyapunov Function Candidate for the Most Generalized Form of the Fourth Order Nonlinear Equation

Consider the fourth order nonlinear differential equation of the form

$$\ddddot{x} + f(\dddot{x}) + g(\ddot{x}) + h(\dot{x}) + j(x) = 0 \quad . \; . \; . \qquad (3.1.4)$$

or the equivalent four system

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = w,\ \dot{w} = -f(\dddot{x}) - g(\ddot{x}) - h(\dot{x}) - j(x) \; . \; . \; . \qquad (3.1.5)$$

Our objective is to construct an integrated equation denoted by V for the four system (2.0.3). So, we consider the constant coefficient fourth order differential equation

$$\ddddot{x} + a\dddot{x} + b\ddot{x} + c\dot{x} + dx = 0 \quad . \; . \; . \quad (3.1.6)$$

or the equivalent four system

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = w,\ \dot{w} = -a\dddot{x} - b\ddot{x} - c\dot{x} - dx \; . \, . \, . \quad (3.1.7)$$

where a, b, c, d are positive constants. We therefore proceed as follows: Multiply equation (4.1.6) by $\dddot{x}$ and integrate with respect to t from $t = 0$ to $t = T$ which yields,

$$\left[\tfrac{1}{2}\dddot{x}^2 + \tfrac{1}{2}d\ddot{x}^2 + dx\ddot{x}\right] + \int_0^T a\dddot{x}^2 ds + \int_0^T c\dot{x}\dddot{x} ds - \int_0^T d\dot{x} ds = 0 \quad . \; . \; . \quad (3.1.8)$$

Let $V = \tfrac{1}{2}\dddot{x}^2 + \tfrac{1}{2}b\ddot{x}^2 + dx\ddot{x}$, so that

$$V + \int_0^T a\dddot{x}^2 ds + \int_0^T c\dot{x}\dddot{x} ds - \int_0^T d\dot{x} ds = 0 \quad . \; . \; . \quad (3.1.9)$$

$$V = -\int_0^T a\dddot{x}^2 ds - \int_0^T c\dot{x}\dddot{x} ds + \int_0^T d\dot{x} ds = 0 \quad . \; . \; . \quad (3.2.0)$$

$$\dot{V} = -a\dddot{x}^2 - c\dot{x}\dddot{x} + d\ddot{x} = 0 \quad . \; . \; . \quad (3.2.1)$$

$$\dot{V} = -\left(a\dddot{x}^2 + c\dot{x}\dddot{x} - d\ddot{x}\right) = 0 \quad . \; . \; . \quad (3.2.2)$$

From equation (1.0.1), that is

$$\dot{V} = -U$$

$$U = a\dddot{x}^2 + c\dot{x}\dddot{x} - d\ddot{x} \quad . \; . \; . \quad (3.2.3)$$

Similarly the derivative of $V = \tfrac{1}{2}\dddot{x}^2 + \tfrac{1}{2}b\ddot{x}^2 + dx\ddot{x}$, along the solution paths of equation (3.1.7) is given by

$$\dot{V} = \dddot{x}\ddddot{x} + b\ddot{x}\dddot{x} + dx\dddot{x} + d\ddot{x} \quad . \; . \; . \quad (3.2.4)$$

$$\dot{V} = \left(-aw - bz - cy - dx\right)w + bzw + dxw + dz \quad . \; . \; . \quad (3.2.5)$$

$$\dot{V} = -aw^2 - bzw - cyw - dxw + bzw + dxw + dz \quad . \; . \; . \quad (3.2.6$$

$$= -aw^2 - cwy + dz \quad . \; . \; . \quad (3.2.7)$$

From equations (3.1.8), (3.1.9), (3.2.0) and (3.2.1), we infer that

$$\dot{V} = -a\dddot{x}^2 - c\dot{x}\dddot{x} + d\ddot{x} \quad . \; . \; . \quad (3.2.8)$$

Therefore, we have gotten a positive definite function

$$V = \tfrac{1}{2}\dddot{x}^2 + \tfrac{1}{2}b\ddot{x}^2 + dx\ddot{x} \quad . \; . \; . \quad (3.2.9)$$

for the system (3.1.4) or (3.1.5) Next, we generalized the V in equation (3.2.9) to the corresponding nonlinear system (3.1.4). However, the comparison between equations (3.1.5) and (3.1.7) shows that equation (3.1.5) is equivalent to equation (3.1.7) if

$f(\dddot{x})$ is replaced by $a\dddot{x}$

$g(\ddot{x})$ is replaced by $b\ddot{x}$

$h(\dot{x})$ is replaced by $c\dot{x}$

$j(x)$ is replaced by dx

Observe that the term 'a' does not appear in equation (3.2.9), which suggest that our V for equation (3.1.5) is defined by

$$V = V(x, y, z, w) = \tfrac{1}{2}\dddot{x}^2 + G(\ddot{x}) + xJ(\ddot{x}) \quad . \; . \; . \quad (3.3.0)$$

where

$$\left.\begin{aligned} G(\ddot{x}) &= \int_0^x g(s)\,ds \\ J(\ddot{x}) &= \int_0^x j(s)\,ds \end{aligned}\right\} \quad . \; . \; . \quad (3.3.1)$$

3.2 Lyapunov Function Candidate for Some Special Form of Fourth Order Nonlinear Differential Equation

Consider

$$\ddddot{x} + f(\dddot{x}) + g(\ddot{x}) + h(t, \dot{x}) + a_4 x = 0 \quad . \; . \; . \quad (3.3.2)$$

$a_4 > 0$ is a constant of the equivalent system

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = w,\ \dot{w} = -f(\dddot{x}) - g(\ddot{x}) - h(t, \dot{x}) - a_4 x \; . \, . \, . \quad (3.3.3)$$

Our concern here is to construct an integrated equation for the four system. So consider the fourth order differential system (2.0.1) or the equivalent four system (2.0.3) in which a, b, c, d are positive constants. Multiply equation (2.0.1) by $\dddot{x}$ and integrate with respect to t, t from [0 to T]. That is

$$\left[\tfrac{1}{2}a\dddot{x}^2 + \tfrac{1}{2}b\ddot{x}\dddot{x} + \tfrac{1}{2}c\dot{x}\dddot{x} + d\ddot{x}^2\right] + \int_0^T \dddot{x}^2 ds - \int_0^T c\dddot{x}^2 ds = 0 \; . \, . \, . \quad (3.3.4)$$

$$V = V(x, y, z, w) = \tfrac{1}{2}a\dddot{x}^2 + \tfrac{1}{2}b\ddot{x}\dddot{x} + \tfrac{1}{2}c\dot{x}\dddot{x} + d\ddot{x}^2 \; . \, . \, . \quad (3.3.5)$$

then the equation (6.2) becomes

$$V+\int_0^T \dddot{x}^2 ds-\int_0^T c\ddot{x}^2 ds=0 \quad . \; . \; . \quad (3.3.6)$$

$$V=-\int_0^T \dddot{x}^2 ds+\int_0^T c\ddot{x}^2 ds \quad . \; . \; . \quad (3.3.7)$$

Differentiating equation (3.3.7) with respect to t gives

$$\dot{V}=-\dddot{x}^2+c\ddot{x}^2$$

$$\dot{V}=-\left(\dddot{x}^2-c\ddot{x}^2\right) \quad . \; . \; . \quad (3.3.8)$$

$$\dot{v}=-u$$

$$u=\dddot{x}^2-c\ddot{x}^2 \quad . \; . \; . \quad (3.3.9)$$

Again differentiating v along the solution paths of equation (2.0.3)

$$v=\tfrac{1}{2}a\ddot{x}^2+\tfrac{1}{2}b\ddot{x}\dddot{x}+\tfrac{1}{2}c\dot{x}\ddot{x}+d\ddot{x}^2$$

this differentiation is done to verify the positive definiteness of v along these solution paths

$$\dot{v}=a\ddot{x}\dddot{x}+\tfrac{1}{2}b\dot{x}\dddot{x}+\tfrac{1}{2}b\ddot{x}\ddot{x}+\tfrac{1}{2}c\ddot{x}\dddot{x}+\tfrac{1}{2}c\dot{x}\ddot{x}+2d\ddot{x} \quad . \; . \; . \quad (3.3.9)$$

$$\dot{v}=aw\left(-aw-bz-cy-dx\right)+\tfrac{1}{2}bw^2+\tfrac{1}{2}bz(-aw-bz-cy-dx)+\tfrac{1}{2}czw+\tfrac{1}{2}cy\left(-aw-bz-cy-dx\right)+2dw$$

$$\dot{v}=-a^2w^2-abwz-acwy-adwx+\tfrac{1}{2}bw^2-\tfrac{1}{2}abwz-\tfrac{1}{2}b^2z^2-\tfrac{1}{2}bczy-\tfrac{1}{2}bdzx+\tfrac{1}{2}czw-\tfrac{1}{2}acwy-\tfrac{1}{2}bczy-\tfrac{1}{2}c^2y^2-\tfrac{1}{2}dcxy+2dw$$

$$2a^2>b,\; 2ab>c, a>2$$

$$\dot{v}=-w^2\left(a^2-\frac{b}{2}\right)-\tfrac{1}{2}z^2b^2-\tfrac{1}{2}y^2c^2-\tfrac{3}{2}wz\left(ab-\frac{c}{2}\right)-\tfrac{3}{2}acwy-\tfrac{1}{2}bdzx-\tfrac{1}{2}dcxy-bczy-wd\left(ax-2\right)$$

Our next work is to generalize v in equation 4.3.5) to the corresponding nonlinear four system (3.3.3). However, the comparison between equation (2.0.1) and (3.3.2)

Shows that equation (2.0.1) is equivalent to equation (3.3.2) if

$f(\ddot{x})$ is replaced by $a\ddot{x}$

$g(\ddot{x})$ is replaced by $b\ddot{x}$

$h(t,\dot{x})$ is replaced by $c\dot{x}$

a_4 is replaced by d

Therefore x

$$V=\int_o^x f(s)ds+\frac{1}{2}g(\ddot{x})\ddot{x}+\frac{1}{2}h(t,\dot{x})\ddot{x}+a_4\ddot{x}^2 \quad . \; . \; . \quad (3.4.1)$$

3.3 OTHER FORMS OF NON-LINEAR FOURTH ORDER DIFFERENTIAL EQUATION

Consider the non linear differential equation

$$\ddot{x}+a\dddot{x}+b\ddot{x}+c\dot{x}+j(x)=0 \quad . \; . \; . \quad (3.4.2)$$

or the equivalent four system

$$\dot{x}=y, \dot{y}=z, \dot{z}=w, \dot{w}=-a\dddot{x}-b\ddot{x}-c\dot{x}-j(x) \quad . \; . \quad (3.4.3)$$

We remark that a close approximation of equation (2.0.3) and (3.4.3) shows that equation (2.0.3) is equivalent to equation (3.4.3). If

$j(x)$ is replaced by dx

It is clearly known that the expression

$$V=\frac{1}{2}\ddot{x}^2+\frac{1}{2}b\ddot{x}^2+dx\ddot{x}$$

is an established integration equation for equation (2.0.3). Therefore,

$$V=\tfrac{1}{2}\ddot{x}^2+\tfrac{1}{2}b\ddot{x}^2+j(x)\ddot{x} \quad . \; . \; . \quad (3.4.4)$$

This corresponds to the four system (4.4.3) as an integrated equation. Furthermore, considering the non linear fourth order equation

$$\ddot{x}+f(\ddot{x})+b\ddot{x}+c\dot{x}+dx=0 \quad . \; . \; . \quad (3.4.5)$$

or the equivalent four system

$$\dot{x}=y, \dot{y}=z, \dot{z}=w, \dot{w}=-f(\ddot{x})-b\ddot{x}-c\dot{x}-dx \quad . \; . \; . \quad (3.4.6)$$

The comparison between equation (2.0.3) and (3.4.6) yields that equation (2.0.3) is equivalent to equation (3.4.6) if

$f(\ddot{x})$ is replaced by $a\ddot{x}$ thus

$$v=\int_o^x f(s)ds+b\ddot{x}^2+dx\ddot{x} \quad . \; . \; . \quad (3.4.8)$$

This corresponds to the four system (3.4.3) as an integrated equation.

4.0 SIGNIFICANCE OF STUDY

The integrated equations or the Lyapunov functions are useful in the study of qualitative properties of ordinary differential equation. Essentially, the study of the hypothesis of an ordinary differential equation that clearly determines whether the solutions are stable, bounded or periodic without actually going through the solutions. Stability, boundedness and periodicity of solutions is indispensable in social sciences, sciences and engineering. Hence, the study of these properties is a useful tool in every technological innovation and is results oriented. We are all concerned with stable economy, stable government, stability of continued energy generation, stability of peace and sound stable ecological system to mention just a few. In engineering the strength of materials is such that can withstand stress and strain as the materials remain stable. Little wonder, stability theory has become a major area of interest to a great number of mathematicians.

5.0 CONCLUSION

The main key in construction of Lyapunov function of non linear differential equation by the method of integration is the fact that corresponding to any positive definite quadratic form $u(x)$, there exists another positive definite quadratic form $v(x)$ such that the time derivative of $v(x)$ along the solution paths of the four scales system is equal to the negative $u(x)$, that is $\dot{V} = -U$

is satisfied; leading us to obtain the Lyapunov type of functions or "integrated equations".

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