On The Error Analysis of The New Formulation of One Step Method Into Linear Multi Step Method For The Solution of Ordinary Differential Equations

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Abstract-This paper presents error analysis of the new formulation of one step method into linear multistep method for the solution of ordinary differential equations. Error analysis in one step method is considerably more difficult than linear multistep method due to the loss of linearity in the method, especially for Runge Kutta Method that makes no mention of the function which defines the differential equation, makes it impossible to define the order of the method, independently of the differential equation. The new formulation of one step method into linear multistep method helps us in overcoming the shortcoming of the method. The error constant, zero stability, consistency and convergence of one step method were determined, through a similar process like linear multistep methods.

Keywords: Convergence, Error, Linear Multistep Method, One Step Method, Runge Kutta Method, Zero Stability

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1 INTRODUCTION

The problem of solving ordinary differential equations is classified into initial value and boundary value problems, depending on the conditions specified at the end points of the domain. There are numerous one step methods for the solution of initial value problems in ordinary differential equations such as Euler's method which was the oldest and simplest method originated by Leonhard Euler in 1768, Improved Euler's method and Runge Kutta methods described by Carl Runge and Martin Kutta in 1895 and 1905 respectively. There are many excellent and exhaustive texts on this subject that may be consulted, such as [2], [5] and [6] just to mention few. The initial problem for a system of first order ordinary differential equations is given by

$$x' = f(t,x), x(t_0) = x_0, t \in [a,b] \quad (1)$$

We shall consider the Runge Kutta method of r stage of the form

$$x_{n+1} = x_n + h\omega(t_n, x_n, h) \quad (2)$$

Where

$$\omega(t_n, x_n, h) = \sum_{j=1}^{r} b_j k_j \quad (3)$$

$$k_j = f(t_n + c_i h, x_n + h\sum_{i=1}^{r} a_{ji} k_i) \quad (4)$$

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Where for $j = 1,2,...,r,$ the real parameters c_i, k_i, a_{ji} define the method and r is the stage number. According to [7], if a matrix is strictly lower triangular i.e. the internal stages can be calculated without depending on later stages, and then the method is called an explicit method otherwise it is called an implicit method. The importance of an implicit method is due to its high orders of accuracy which is superior to the explicit methods. This makes it more suitable for solving stiff problems. In this work we present error analysis of the new formulation of one step method (Runge Kutta Method) into linear multistep method for the solution of ordinary differential equations.

2 LINEAR MULTISTEP METHOD

The general linear multistep method is defined by

$$\sum_{j=0}^{k} \alpha_j y_{n+1} = h\sum_{j=0}^{k} \beta_j f_{n+j} \quad (5)$$

Where α_j and β_j are constants, h is the step size.

2.1 ANALYSIS OF BASIC PROPERTIES OF THE LINEAR MULTISTEP METHOD

The basic properties of numerical method are of order of accuracy, consistency, zero stability and convergence. These are examined for the derived method**.**

2.1.1 ORDER AND ERROR TERM

A characteristic of discretization method is that errors are generated when they are adopted for the solution of ordinary differential equations. Consequently, these errors have to be examined to ensure the approximate value does not deviate significantly from the exact solution as the iteration progresses. The magnitude of these errors is referred to as global error denoted by

$$e_n = y(x_n) - y_n \quad (6)$$

Where $y(x_n)$ and y_n are the exact and computed solution at x_n respectively. This is supposed to be small since the magnitude of the error determines the accuracy of the scheme. The linear multistep method (5) is said to be of order p if $c_0 = c_1 = ... = c_p = 0$ but $c_{p+1} \neq 0$ and c_{p+1} is called the error constant, where

$$c_q = \sum_{j=0}^{k} \alpha_j \frac{j^q}{q!} - \sum_{j=0}^{k} \beta_j \frac{j^{(q-1)}}{(q-1)!} \tag{7}$$

2.1.2 CONSISTENCY [5, 6]

Equation (5) is said to be consistent if it can satisfy the following conditions:

i. The order of the scheme must be greater than or equal to one

ii. $\sum_{j=0}^{k} \alpha_j = 0$

iii. $\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$ (8)

iv. $P(1) = 0$

v. $P'(1) = \sigma(1)$

2.1.3 ZERO STABILITY

According to [5, 6] a linear multistep method is said to be zero stable if no root of the first characteristic polynomial $P(v)$ has modulus greater than one and if every root of modulus one is simple.

2.1.4 CONVERGENCE

The necessary and sufficient conditions for a linear multistep to convergence are zero stability and consistency [3] with the addition of the hypothesis

$$\lim \left(\frac{\eta^h - \eta^0}{h} \right) = \hat{\eta} \tag{9}$$

2.2 THE NEW FORMULATION

We shall consider the classical Runge Kutta method of order four, order five and sixth order implicit Runge Kutta method as follows:

2.2.1 CLASSICAL RUNGE KUTTA METHOD OF ORDER FOUR [4]

The Runge Kutta method of order four is given by

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{10}$$

Where

$$k_1 = f(x_n, y_n),\ k_2 = f(x_n + 0.5h, y_n + 0.5k_1),$$
$$k_3 = f(x_n + 0.5h, y_n + 0.5k_2),$$
$$k_4 = f(x_n + h, y_n + k_3), \tag{11}$$

We shall let

$$k_1 = f_{c_1}, k_2 = f_{c_{21}}, ..., k_s = f_{c_s} \tag{12}$$

From equation (11), we have $c_1 = 0, c_2 = c_3 = \frac{1}{2}$ and $c_4 = 1$ then, applying (12), we have $k_1 = f_0, k_2 = k_3 = f_{\frac{1}{2}}$ and $k_4 = f_1$, thus (10) yields

$$y_{n+1} = y_n + \frac{1}{6}\left(f_0 + 4f_{\frac{1}{2}} + f_1 \right) \tag{13}$$

2.2.1 RUNGE KUTTA METHOD OF ORDER FIVE

The well-known fifth order Runge Kutta method is given by [1]

$$y_{n+1} = y_n + \frac{h}{192}(23k_1 + 125k_3 - 81k_5 + 126k_6) \tag{14}$$

Where

$$k_1 = f(x_n, y_n), \quad k_2 = f(x_n + \tfrac{1}{3}h, y_n + \tfrac{1}{3}k_1),$$
$$k_3 = f(x_n + \tfrac{2}{5}h, y_n + \tfrac{h}{25}(4k_1 + 6k_2)), \tag{15}$$
$$k_4 = f(x_n + h, y_n + \tfrac{h}{4}(k_1 - 12k_2 + 15k_3)),$$
$$k_5 = f(x_n + \tfrac{2}{3}h, y_n + \tfrac{h}{81}(6k_1 + 90k_2 - 50k_3 + 8k_4))$$
$$k_6 = f(x_n + \tfrac{4}{5}h, y_n + \tfrac{h}{75}(6k_1 + 36k_2 + 10k_3 + 8k_4))$$

From equation (15), we have

$c_1 = 0, c_2 = \frac{1}{3}, c_3 = \frac{2}{5}, c_4 = 1, c_5 = \frac{2}{3}$ and $c_6 = \frac{4}{5}$ then, applying (12), we have

$k_1 = f_0, k_2 = f_{\frac{1}{3}}, k_3 = f_{\frac{2}{3}}, k_4 = f_1, k_5 = f_{\frac{2}{3}}$ and $k_6 = f_{\frac{4}{5}}$, thus (14) yields

$$y_{n+1} = y_n + \frac{h}{192}\left(23f_0 + 125f_{\frac{2}{5}} - 81f_{\frac{2}{3}} + 125f_{\frac{4}{5}} \right) \tag{16}$$

3 THE ERROR ANALYSIS OF THE NEW FORMULATION

Now we shall investigate the error analysis of the new formulation of the fourth and fifth orders Runge Kutta method by considering the order, error constant and zero stability [1]. From equations (5) and (7), we have for the classical Runge Kutta method of order four in equation (13)

$c_1 = c_2 = c_3 = c_4 = 0$, but $c_{p+1} = c_5 = 3.472 \times 10^{-4}$.

Also we have for new fifth order Runge Kutta method on equation (16) that

$c_1 = c_2 = c_3 = c_4 = c_5 = 0$,

but $c_{p+1} = c_6 = 1.852 \times 10^{-5}$.

Hence from the definitions in 2.1.3 and 2.1.4, equations (13) and (16) are zero stable and convergent.

4 CONCLUSION

This approach reduces computational efforts in determining the order and convergence of Runge Kutta method. It also allows arbitrary higher order to be formulated. The new formulation of Runge Kutta method into linear multistep method is consistent, zero stable and convergent.

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