Crank-Nicholson -Lax-Friedrich's Finite Difference Schemes Arising From Operator Splitting For Solving 2-Dimensional Heat Equation

John K. Rotich, Simeon K. Maritim, Jakob K. Bitok

Abstract: We develop hybrid finite difference schemes arising from operator splitting to solve 2-D heat equations. We developed the Crank-Nicholson-Lax-Fredrick's hybrid scheme and determine that the method is more accurate than pure Crank-Nicholson method. The method is unconditionally stable because it is Crank-Nicholson based.

Index Terms: Crank-Nicholson, Finite Difference Schemes, Lax-Friedrich, Operator Splitting

1 INTRODUCTION

The 2-D parabolic equations are applicable in science, engineering and mathematics. They are also used to describe heat and fluid movement in two directions. So far the methods that have been used to solve such equations are: Finite difference methods (FDM), Alternative Direction Implicit (ADI) methods and locally one dimensional method. Peaceman and Rachford [13] explained that in mathematics, the alternating direction implicit (ADI) method is a finite difference method for solving parabolic and elliptic partial differential equations. It is mostly used to solve the problems of heat conduction for solving the diffusion equation in two or more dimensions. The idea behind the ADI method is to split the finite difference equations into two, one with the x-derivative taken implicitly and the next with the y-derivative taken implicitly. The systems of equations involved are symmetric and tridiagonal (banded with bandwidth 3), and thus cheap to solve. It has been shown that this method is unconditionally stable. There are more refined ADI methods such as the methods of Douglas [3], or the f-factor method (Chang [2]) which can be used for three or more dimensions. Koller [9] and Hochbruck and Osterman [6] demonstrate and discuss time integration due to operator splitting for linear 1-D parabolic equations.

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Ames [1] and Mitchel and Griffiths [12] describes additive operator splitting for parabolic equation which are more than one dimensional and were developed by Yanenko and Marchuk. Another splitting method mentioned by the same author (Mitchel and Griffiths [12]) which is called second order was developed by Strang in the 1960s. Istvan [8] gives an elaborate discussion of operator splitting for parabolic equations. Le Veque and Oliger [11] describes additive operator splitting for hyperbolic partial difference equations. Splitting method has been used by Evje and Hvistendahl [4] to find the numerical solution of convection–diffusion equation.Galligani [5] reviews various additive operators splitting method for solving on parallel computers a large class of semi-discrete diffusion problems. Hvistendahl and Risebro [7] in their paper presented a semi-discrete method for constructing approximate solution of a many dimensional convection diffusion equation. Koross et al [10] solved the 1-D heat equation using operator splitting by modifying it. They developed hybrid finite difference method resulting from operator splitting for solving the modified form. In their paper they proved that there is an improvement in efficacy of the Crank-Nicholson scheme when the Lax-Friedrich's and Du Fort and Frankel discretizations are used on it. They concluded in their research findings that the Crank-Nicholson-Lax-Friedrich-Du For and Frankel is the most accurate method for solving 1-D heat equation. In this paper we apply Koross' [10] work to develop hybrid finite difference schemes arising from operator splitting that can be used to find numerical solution of 2-D heat equation. Among the methods to be developed are: Crank-Nicholson-Du Fort and Frankel, Crank-Nicholson-Lax-Friedrichs, Crank-Nicholson-Du Fort and Frankel-Lax Friedrich's. We also develop the pure Crank-Nicholson scheme. This will serve to provide a good comparison of two and three level schemes. We organize this paper as follows: in section 2 we outline operator splitting, in section 3 we develop hybrid finite difference schemes and in section 4 we present and discuss the results.

2 OPERATOR SPLITTING

We consider the parabolic equation

$u_t = \alpha u_{xx} + \beta u_{yy}, (0 \le x, y \le a) \times (t \ge 0)$ (2.1)

$u(x, y, 0) = u_0(x, y)$ (2.2)

where $u = u(x, y, t)$.

Koross et al [10] gave the outline of operator splitting for 1-D parabolic equation as

$U_{m,n+1} = \prod_{i=1}^{s} e^{kL_i} U_{m,n}$ (2.3)

We introduce another spatial direction in equation (2.3) and so we get

$U_{m,l,n+1} = \prod_{i=1}^{s} e^{kL_i} U_{m,l,n}$ (2.4)

The approximate solution can be obtained from equation (2.3) by first solving

$$U_{m,l,n+1}^{(s)} = e^{kL_s} U_{m,l,n}$$

and then using this solution we can find

$$U_{m,l,n+1}^{(s-1)} = e^{kL_{s-1}} U_{m,l,n}$$

We go on like this until we attain

$$U_{m,l,n+1}^{(1)}$$

Which is actually the approximate solution of equation (2.1). The approximate solution of (2.3) is found by

$$\begin{aligned} U_{m,l,n+1} &= e^{kL_1}\left(e^{kL_2} U_{m,l,n}\right)) \quad (2.5) \\ &= \left(1 + kL_1 + \frac{1}{2}k^2L_1^2 + \cdots\right)\left(1 + kL_2 + \frac{1}{2}k^2L_2^2 + \cdots\right)U_{m,l,n} \\ &= \left(1 + kL_1 + kL_2 + k^2L_1L_2 + \frac{1}{2}k^2L_1^2 + \frac{1}{2}k^2L_2^2 + \frac{1}{2}k^3L_1L_2^2 \right. \\ &\quad \left. + \frac{1}{2}k^3L_1^2L_2 + \frac{1}{4}k^4L_1^2L_2^2 + 0(k^5)\right) U_{m,l,n} \end{aligned}$$

$$\approx \left(1 + kL_1 + kL_2 + k^2L_1L_2 + \tfrac{1}{2}k^2L_1^2 + \tfrac{1}{2}k^2L_2^2 + \tfrac{1}{2}k^3L_1L_2^2 + \tfrac{1}{2}k^3L_1^2L_2\right) U_{m,l,n} \quad (2.6)$$

We organize this paper as follows: in section 2 we outline operator splitting and develop hybrid finite difference schemes and in section 3, we present and discuss the results.

3DEVELOPMENT OF THE HYBRID SCHEMES

3.1 Pure Crank-Nicholson (CN) scheme

We consider the 2-D heat equation

$$U_t = \alpha U_{xx} + \beta U_{yy}, (0 \le x, y \le 1) \times (t \ge 0)$$

$$u(x, y, 0) = \sin \pi x \sin \pi y \qquad (3.1.1)$$

Here

$$s = 2$$

and so

$L = L_1 + L_2$ where $L_1 = \frac{\partial^2}{\partial x^2} \approx \frac{1}{h^2}\delta_x^2$ and $L_2 = \frac{\partial^2}{\partial y^2} \approx \frac{1}{q^2}\delta_y^2$

It is necessary that we first develop the pure Crank-Nicholson method resulting from this splitting. This is because other hybrid methods are derived from it. Thus the Crank-Nicholson method is as follows:

$$L_1 U_{m,l,n} = \frac{\alpha}{4h^2}\delta_x^2 \left(U_{m,l,n} + U_{m,l,n+1}\right) \qquad (3.1.2)$$

$$L_2 U_{m,l,n} = \frac{\beta}{4q^2}\delta_y^2 \left(U_{m,l,n} + U_{m,l,n+1}\right) \qquad (3.1.3)$$

$$L_1 L_2 U_{m,l,n} = \frac{\alpha\beta}{16h^2q^2}\delta_x^2 \left(\delta_y^2\left(U_{m,l,n} + U_{m,l,n+1}\right)\right) \qquad (3.1.4)$$

$$L_2^2 U_{m,l,n} = \frac{\beta^2}{16q^4}\delta_y^2\left(\delta_y^2\left(U_{m,l,n} + U_{m,l,n+1}\right)\right). \qquad (3.1.5)$$

$$L_1^2 U_{m,l,n} = \frac{\alpha^2}{16h^4}\delta_x^2\left(\delta_x^2\left(U_{m,l,n} + U_{m,l,n+1}\right)\right) \qquad (3.1.6)$$

$$L_1 L_2^2 U_{m,l,n} = \frac{\alpha\beta^2}{64h^2q^4}\delta_x^2\left(\delta_y^2\left(\delta_y^2\left(U_{m,l,n} + U_{m,l,n+1}\right)\right)\right) \qquad (3.1.7)$$

and

$$L_1^2 L_2 U_{m,l,n} = \frac{\alpha^2\beta}{64h^4q^2}\delta_x^2\left(\delta_x^2\left(\delta_y^2\left(U_{m,l,n} + U_{m,l,n+1}\right)\right)\right) \qquad (3.1.8)$$

Using equations (3.1.2) - (3.1.8) in equation (2.5) we have

3.2 Crank-Nicholson -Lax-Friedrich's (CN-LF) scheme

For this scheme the first term $U_{m,l,n}$ in the right hand side of equation (2.5) is replaced by

$$\frac{1}{2}\left(U_{m+1,l+1,n} + U_{m-1,l+1,n}\right) + \frac{1}{2}\left(U_{m+1,l-1,n} + U_{m-1,l-1,n}\right)$$

4 RESULT OF THE NUMERICAL EXPERIMENTS

We present the results using the following data: $k = 0.0001, h = 0.25, l = 0.25, \alpha = 1, \beta = 1$, $u(x, y, 0) = \sin\pi x\sin\pi y$ and $u(x, y, t) = e^{-\pi^2 t}\sin\pi x\sin\pi y$ We now present the results. We shall display these results using graphs, tables as well as 3-D figures.

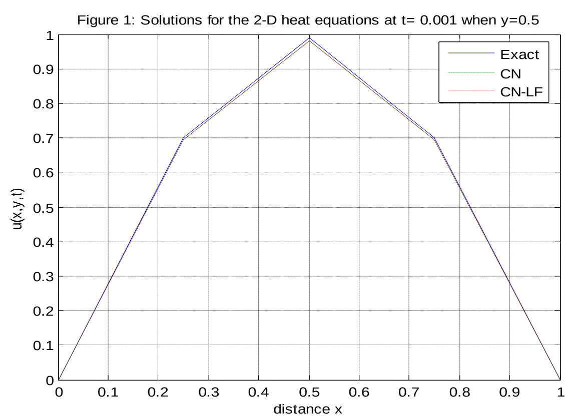

Figure 1: *Solutions for the 2-D heat equation from the different methods at t = 0.001 when y=0.5*

As expected the value of the solution is maximum when x = 0.5. This is because the analytic solution $u(x, y, t) = e^{-\pi^2 t}\sin\pi x\sin\pi y$ is maximum when $\pi x = \pi y = \frac{\pi}{2}$ (that is at: $x = y = 0.5$). The graph is not smooth because the value of h is not small enough. When h is small enough the curve will be a smooth parabola. Table 1 below provides the solution of 2-D heat equation for the different schemes developed.

***Table 1**: Solutions of the 2-D heat equations at t = 0.001 for the different schemes*

x	y	Exact	CN	CN-LF
0	0	0	0	0
0.2	0.2	0.495089470153	0.490703392994	0.490703392994
5	5	74	62	64
	0.2	0.700162243279	0.693959393475	0.693959393475
0.5	5	52	50	50
0.7	0.2	0.495089470153	0.490703392994	0.490703392994
5	5	74	62	64
	0.2			
0	5	0	0	0
0.2		0.700162243279	0.693959393475	0.693959393475
5	0.5	52	50	52
		0.990178940307	0.981406785989	0.981406785989
0.5	0.5	47	29	29
0.7		0.700162243279	0.693959393475	0.693959393475
5	0.5	52	50	52
0	0.5	0	0	0
0.2	0.7	0.495089470153	0.490703392994	0.490703392994
5	5	74	62	64
	0.7	0.700162243279	0.693959393475	0.693959393475
0.5	5	52	50	50
0.7	0.7	0.495089470153	0.490703392994	0.490703392994
5	5	74	62	64
	0.7			
0	5	0	0	0

The Figure 1 and Table 1 does not provide a clear comparison because the curves almost coincide and the numbers are almost equal respectively. We provide a table of absolute errors to give us a clear comparison. This is done in Table 2.

***Table 2**: Absolute errors in solution of 2-D heat equation from different schemes at t = 0.001*

x	y	CN	CN-LF
0	0	0	0
0.25	0.25	0.00438607715912	0.00438607715910
0.5	0.25	0.00620284980402	0.00620284980402
0.75	0.25	0.00438607715912	0.00438607715910
0	0.25	0	0
0.25	0.5	0.00620284980402	0.00620284980400
0.5	0.5	0.00877215431818	0.00877215431818
0.75	0.5	0.00620284980402	0.00620284980400
0	0.5	0	0
0.25	0.75	0.00438607715912	0.00438607715910
0.5	0.75	0.00620284980402	0.00620284980402
0.75	0.75	0.00438607715912	0.00438607715910
0	0.75	0	0

From Table 2 we can tell that hybrid CN-LF provides the most accurate results because it produces the least absolute error.

We now present 3-D solutions:

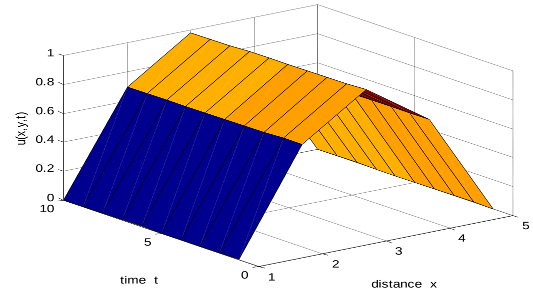

***Figure 2**: CN solution for the 2-D heat equation when y = 0.5*

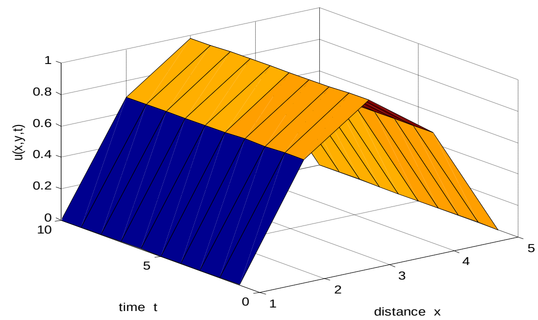

***Figure 3**: CN-LF solution for the 2-D heat heat equation at y = 0.5*

At any given value of t the solution is a parabola as that of Figure 1. We note that the 3-D solutions from all the methods developed take the same shape.

5 Conclusion

We have established that hybrid Crank-Nicholson-Lax-Friedrich's scheme is the more accurate method compared to the Pure Crank-Nicholson method. There is an improvement of the efficacy of the Crank-Nicholson scheme when Lax-Friedrich's discretization is used on it. The increase of grid points involved is responsible for the improved accuracy. Since the hybrid methods is Crank-Nicholson based, the schemes developed is unconditionally stable. The method developed can be applied to any other 2-D parabolic equations.

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APPENDIX: ACRONYMS AND ABBREVIATIONS

The following notations are used throughout the presentations;
CN- Pure Crank-Nicholson
CN-LF- Crank-Nicholson-Lax-Friedrich's
2-D -Two dimensional
3-D-Three dimensional
δ – Central difference operator