On General Product Of Two Finite Cyclic Groups One Being Of Order 5 (Induced By π = (1) (2) (3) (4) (5))

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Abstract: In this paper we find the general product induced by the semi special permutation $\pi = (1)(2)(3)(4)(5)$. That is the general products of two finite cyclic groups in which one of order 5 and the other is of order m these general products can be described in terms of numerical parameters.

Index Terms: semi special permutations, general product.

1 INTRODUCTION

If *A, B* are two subgroups of a group*G* then we say that *G* is the general product of *A, B* if and only if:

(1) *G = AB*

(2) *A, B* has no elements in common other than the identity i.e. $A \cap B = \{e\}$.

Now if*A = {a}* is a cyclic group of order *m, B = {b}* is a cyclic group of order n then there exist corresponding to *G* two semi special permutations π, ρ where π on *[n]*, ρ on *[m]*such that

$$a^y b^x = b^{\pi^y x} a^{\rho^x y}, x \in [n], y \in [m] \quad \text{.... (1)}$$

$$\pi^m x \equiv x \,(mod\ n), x \in [n] \quad \text{.....(2)}$$

$$\rho^n y \equiv y \,(\text{mod}\, m), y \in [m] \quad \text{.....(3)}$$

Where *[c]* demote to the set of dements *{1, 2, 3, …, c}*

Definition:(Semi special permutation)*A* permutation π on *[c]* is said to be semi special on *[c] iff* π *(c) = c*,

$\pi_z(x) = \pi(x+z) - \pi z$ (mod*c*), y $\in$ [c] is a power depending on *z* of π

Theorem A:

(i) $a^m b^x = b^x a^m, x \in [n]$ …….. (4)

(ii) $a^y b^n = b^n a^y, y \in [m]$ …….. (5)

Theorem B:

(i) The order of π divides *m* i.e. if e is the orders of π then m is a multiple of ℓ.

(ii) There exist a number $\lambda, (\lambda, \frac{m}{\ell}) = 1$ thus that

$$a^{\ell} b = b a^{\ell\lambda}, \ell\lambda^{\mu} \equiv \ell \,\text{mod}\, m \quad \text{……. (6)}$$

Where μ is the **g.c.d** of all $\nu - \upsilon; \nu, \upsilon$ are any numbers of the principal cycle of π.

(iii) $a^{\ell} b^{\mu} = b^{\mu} a^{\ell}$.

We know that $\pi =$ (1) (2) (3) (4) (5) is a semi special permutation on[5]

§1- The general product induced by π

Theorem1.1: the defining relation of the general product of*G* corresponding to

$$\pi = (1)(2)(3)(4)(5) \text{ is } G = \{a, b; a^m = b^5 = e, ab = ba^r\} \quad \text{…… (7.1)}$$

$$r^5 \equiv 1 (\text{mod}\, m) \quad \text{…….. (7.2)}$$

The converse is also true i.e. for any *r* satisfying (7.2) then any group *G* generated by *a* and*b* satisfying (7.1) is the general product of {*a*}, {*b*}.

Proof:

Assume that the general product of G exist, from the equation

$a^y b^y = b^{\pi^y x} a^{\rho^x y}$, $x \in [5], y \in [m]$, with *y*=1 we get

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$ab^{x} = b^{\pi 1 x} a^{\rho^{x} 1}, x = 1,2,3,4,5$ put x=1 then we have

$ab = b^{\pi 1} a^{\rho 1}$

let us write $\rho 1 = r$ then $ab = ba^{r}$,

$$ab^{2} = abb = ba^{r} b = b\underbrace{aaa...a}_{r\text{-times}}b$$

$ab^{2} = b^{2} a^{r^{2}}$ and so by induction we get $ab^{5} = b^{5} a^{r^{5}}$ ….. (8)

From theorem A with n = 5, y = 1 we have $ab^{5} = b^{5} a$ .... (9)

from8, (9) we get $r^{5} \equiv 1 (\mathrm{mod}\, m)$ and so (7.2) follows.

Also we notice that {a} is of order m and {b} is of order 5 then 7.1 is the required defining relation of G.

The converse is also true to do this let G be a group generated by a, b with the defining relation (7.1) and satisfying the condition (7.2) and let x= {0, 1, 2, 3, 4}, y={0, 1, 2, …, m-1} and let H be the set of all ordered pairs (x, y) with $x \in X, y \in Y$ with binary operation $*$ defined on H as follows:

$(x, y) * (x', y') = (x'', y'')$ such that $x'' = x + x' \,(\mathrm{mod}\ 5)$

$y'' = r^{x'} y + y' \,(\mathrm{mod}\ \mathrm{m})$

Then it is clear that $<H, *>$ is a group with e=(0,0) as its identity element. Also if $\alpha = (0,1), \beta = (1,0)$

$\beta^{x} \alpha^{y} = (x, y)$ which implies that each element of H can be determined uniquely in the form $\beta^{x} \alpha^{y}$ which means that H is the general product of $\{\alpha\},\{\beta\}$. since $\{\alpha\}$ is of order m and $\{\beta\}$ is of order 5 so $|H| = 5m$,it is evident to see that $\alpha^{m} = \beta^{m} = e$, $\alpha\beta = \beta\alpha^{r}$ which arecorresponding to the defining relation of G and so the permutation $\pi = (1)(2)(3)(4)(5)$ is induced by α .

Also H can be considered as a homomorphic image of G, since $|G| \leq 5m$ and hence the two groups are isomorphic hence the theorem is proved.

Remark: It must be noted that two groups G, L with defining relation:

$G = \{a, b; a^{m} = b^{5} = e, ab = ba^{r}, r^{5} \equiv 1 (\mathrm{mod}\, m)\}$

$L = \{a, b; a^{m} = b^{5} = e, ab = ba^{s}, s^{5} \equiv 1 (\mathrm{mod}\, m)\}$

Such that $r \not\equiv s$ mod m, then $G \cong L$ if and only if $r \equiv s^{4} (\mathrm{mod}\, m)$

Conclusion:

The general product of two finite cyclic groups one being of order 5, which is corresponding to $\pi = (1)(2)(3)(4)(5)$ is obtained by theorem 1.1 with defining relation (7.1), (7.2).

References:

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