INTERNATIONAL JOURNAL OF SCIENTIFIC & TECHNOLOGY RESEARCH VOLUME 2, ISSUE 5, MAY 2013 ISSN 2277-8616

# Moyal Lax Integrability Of Standard And Pseudo Hierarchy Equations

A. EL Boukili, M. Nach, M. B. Sedra

**Abstract**: In This paper we will study Non commutative $sl_n$-KdV hierarchy equations by the means of Moyal Non commutative Lax generating technics. We will make some consistent assumptions shown to be essential in deriving the Lax pair of special noncommutative integrables systems, for this we calculate positive and negative orders of Moyal Noncommutative hierarchy equations for Burgers and KdV systems. We present also the conditions of having linear equations and giving the ansatz Process and the NC hierarchy equations in the general case.

**Index Terms**: Moyal noncommutativity, integrability, Lax equations, Burgers equation, KdV equations

## 1. INTRODUCTION

More recently there has been a growth in the interest in non-commutative geometry (NCG), which appears in string theory in several ways [1]. Much attention has been paid also to field theories on NC spaces and more specifically Moyal deformed space-time, because of the appearance of such theories as certain limits of string, D-brane and M-theory [2],[3]. Non-commutative field theories emerging from string (membrane) theory stimulate actually a lot of important questions about the non-commutative integrables systems and how they can be described in terms of star product and Moyal bracket [4],[5]. Recall that in the Moyal momentum algebra the ordinary pseudo differential Lax operators [6] and [7]

$$L_n = \sum_{j\in Z} u_{n-j}\,\partial^j, \qquad (1)$$

is naturally replaced by the Moyal momentum Lax operators

$$\mathcal{L}_n = \sum_{j\in Z} u_{n-j} * p^j, \qquad (2)$$

This work is presented as follows: We give in *section 2* our convention notations with a recall of the basic lines of the Moyal momentum algebra. *Section 3* is devoted to a set up of the Lax pair representation in higher order of special integrable systems namely the KdV and Burgers modeles. In section 4 we gives the (Non standard) pseudo formulation of the Lax generating technics applied to Burgers systems. *section 5* is for conclusion.

- Corresponding Author: Abderrahman EL Boukili
- E-mail: aelboukili@gmail.com
- Ibn Tofail University, Faculty of Sciences, Physics Departement, (LHESIR), Kenitra, Morocco.

## 2. ON THE MOYAL MEMENTUM ALGEBRA

### 2.1 Algebric Structure

This is the algebra based on arbitrary momentum differential operators of arbitrary conformal weight m and arbitrary degrees (r,s). Its obtained by summing over all the allowed values of spin (conformal weight) and degrees in the following way[6]-[10]

$$\Sigma(\theta) = \oplus_{r\le s}\oplus_{m\in Z}\Sigma_m^{(r,s)} \qquad (3)$$

with $\Sigma_m^{(r,s)}$ is the space of momentum differential operators of conformal weight m and degrees (r,s) with $r \le s$. Typical operators of this space are given by

$$\mathcal{L}_m^{(r,s)}(u) = \sum_{i=r}^{s} u_{m-i}(x) * p^i \qquad (4)$$

Noting $\Sigma_m^{(k,k)}$ the unidimensional subspaces containing prototype elements of kind $u_{m-k} * p^k$ or $p^k * u_{m-k}$. Using the $\theta$ -Leibniz rule, we can write, for fixed value of k:

$$\Sigma_m^{(k,k)} \equiv \hat{\Sigma}_m^{(k,k)} \oplus \theta\hat{\Sigma}_m^{(k-1,k-1)} \oplus \theta^2\hat{\Sigma}_m^{(k-2,k-2)} \oplus \dots \qquad (5)$$

where $\hat{\Sigma}_m^{(k,k)}$ is the standard one dimensional sub-space of Laurent series objects $u_{m-k}p^k$ considered also as the $(\theta = 0)$ -limit of $\Sigma_m^{(k,k)}$.

We can extract from the space $\Sigma_m^{(r,s)}$ the subalgebra $\Sigma_0^{(-\infty,1)}$ which have the remarkable space decomposition

$$\Sigma_0^{(-\infty,1)} = \Sigma_0^{(-\infty,-1)} \oplus \Sigma_0^{(0,1)}, \qquad (6)$$

where $\Sigma_0^{(-\infty,-1)}$ describes the Lie algebra of pure non local momentum operators and $\Sigma_0^{(0,1)}$ is the Lie algebra of local Lorentz scalar momentum operators $L_0(u) = u_{-1} * p + u_0$. The latter can splits as follows

$$\Sigma_0^{(0,1)} = \Sigma_0^{(0,0)} \oplus \Sigma_0^{(1,1)}, \tag{7}$$

where $\Sigma_0^{(1,1)}$ is the Lie algebra of vector momentum fields $J_0(u) = u_{-1} * p$ which are also elements of $\Sigma_0^{(0,1)}$. Forgetting about the fields (of vanishing conformal spin) belonging to $\Sigma_0^{(0,0)}$ is equivalent to consider the coset space

$$\Sigma_0^{(1,1)} \equiv \Sigma_0^{(0,1)} / \Sigma_0^{(0,0)} \tag{8}$$

one obtain the Diff($S^1$) momentum algebra of vector fields $J_0(u) = u_{-1} * p$ namely

$$\{J_0(u), J_0(v)\} = J_0(w) \tag{9}$$

with $w_{-1} = u_{-1}v'_{-1} - u'_{-1}v_{-1}$.

The extension of these results to non local momentum operators is natural. In fact, one easily show that the previous Lie algebras are simply sub-algebras of the huge momentum space $\Sigma_0^{(-\infty,1)}$. For a given $0 \le k \le 1$, we have

$$\Sigma_0^{(0,1)} \subset \Sigma_0^{(-\infty,k)} \subset \Sigma_0^{(-\infty,1)} \tag{10}$$

and by virtue of (4)

$$\left[\Sigma_0^{(-\infty,k)}, \Sigma_0^{(0,1)}\right] \subset \Sigma_0^{(-\infty,k)} \subset \Sigma_0^{(-\infty,1)} \tag{11}$$

and for $-\infty < p \le q \le 1$

$$\left[\Sigma_0^{(-\infty,p)}, \Sigma_0^{(-\infty,q)}\right] \subset \Sigma_0^{(-\infty,p+q-1)} \tag{12}$$

These Moyal bracket expressions show in turn that all the subspaces $\Sigma_0^{(p,q)}$ with $-\infty < p \le q \le 1$ are ideals of $\Sigma_0^{(-\infty,1)}$.

**2.2 Moyal Bracket**

The Moyal bracket may be thought of as a deformation of the Poisson bracket by the introduction of higher order derivative terms. It turns out that the Jacobi identity is highly restrictive as to the nature of these terms, and one is lead uniquely to the Moyal bracket:

$$\{f,g\}_\theta = \sum_{s=0}^{\infty} \frac{\theta^{2s}}{(2s+1)!} \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} (\partial_x^j \partial_p^{2s+1-j} f)(\partial_x^{2s+1-j} \partial_p^j g) \tag{13}$$

It has all the standard properties one would expect of such a bracket:

$$\begin{aligned} &\{f,g\}_\theta = -\{f,g\}_\theta, && : \textit{antisymmetry}, \\ &\{af + bg, h\}_\theta = a\{f,h\}_\theta + b\{g,h\}_\theta, && : \textit{linearity}, \\ &\{f,\{g,h\}_\theta\}_\theta + \{g,\{h,f\}_\theta\}_\theta + \{h,\{f,g\}_\theta\}_\theta = 0 && : \textit{Jacobi identity} \end{aligned} \tag{14}$$

(where a,b are independent of x and p). Moreover it has the important property that

$$\lim_{\theta\to 0} \{f,g\}_\theta = \{f,g\}, \tag{15}$$

where the bracket $\{f,g\}$ is just the Poisson bracket

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial x} \tag{16}$$

In the limit as $\theta \to 0$ the bracket collapses to the Poisson bracket (14). It also has many other interesting properties, amongst which is the fact that it may be written in terms of an associative *-product defined by

$$f * g = \sum_{s=0}^{\infty} \frac{\theta^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} (\partial_x^j \partial_p^{s-j} f)(\partial_x^{s-j} \partial_p^j g) \tag{17}$$

This has the property that $\lim_{\theta\to 0} f * g = fg$, and with this the Moyal bracket takes the form:

$$\begin{aligned} \{f,g\}_\theta &= \frac{f*g - g*f}{2\theta} \\ &= \frac{1}{2\theta} \sum_{s=0}^{\infty} \frac{\theta^s}{s!} \sum_{j=0}^{s} \left((-1)^j - (-1)^{s-j}\right) \binom{s}{j} (\partial_x^j \partial_p^{s-j} f)(\partial_x^{s-j} \partial_p^j g) \end{aligned} \tag{18}$$

where $f^{(s)} = \partial_x^s f$ is the prime derivative, $\binom{n}{p} = C_n^s = \frac{n!}{s!(n-s)!}$ and $\theta_{ij}$ is a constant antisymmetric tensor. This definition of the star operator may be extended to a 2N dimensional phase space with canonically conjugate variables $(x_j, p_j)$ as follows

$$f * g = \sum_{j=0}^{N} \sum_{s=0}^{\infty} \frac{\theta^s}{s!} \sum_{i=0}^{s} (-1)^i \binom{s}{i} \left[\partial_{x_j}^{s-i} \partial_{p_j}^{i} f\right]\left[\partial_{x_j}^{i} \partial_{p_j}^{s-i} g\right] \tag{19}$$

**2.3 Moyal Lax Formulation**

The algebra $sl_n \equiv \Sigma_n^{(0,n)}$ describes simply the coset space $\Sigma_n^{(0,n)}/\Sigma_n^{(-1,n-1)}$ of $sl_n$ -Lax operators given by

$$\mathcal{L}_n(u) = p^n + \sum_{i=0}^{n-2} u_{n-i} \star p^i \qquad (20)$$

where we have set $u_0 = 1$ and $u_1 = 0$. This is a natural generalization of the well known $sl_2$ -momentum Lax operator

$$\mathcal{L}_2 = p^2 + u_2 \qquad (21)$$

associated to the $\theta$-KdV integrable hierarchy that we will discuss later. Such operator can be writen as

$$\mathcal{L}_2 = (p + \phi')\star(p - \phi') \qquad (22)$$

where $\phi$ is a Lorentz scalar field.

The first nontrivial $\theta$-deformed Toda field theory is the one associated to the $sl_3$ momentum Lax operator

$$\mathcal{L}_3 = p^3 + u_2 p + w_3, \qquad (23)$$

with $w_3 = u_3 - \theta u_2'$. It reads in the Miura transformation as

$$\mathcal{L}_3 = (p + \phi_1')\star(p + \phi_2')\star(p - \phi_1' - \phi_2') \qquad (24)$$

where $\phi_k$ are Lorentz scalar fields $[\phi_k] = 0$.

Let us note that the equation (21), which describes the Miura transformation in the Moyal Momentum algebra, presents a mapping between the KdV system of conformal weight 2 and the Burgers system of conformal weight 1, and the equation (23) describes the mapping between a system of conformal weight 3 (Boussinesq) and the Burgers system. these two mapping can be schematized as follows[9]:

$$\Sigma_2^{(0,2)}/\Sigma_2^{(-1,1)} \cong \Sigma_1^{(0,1)} \otimes \Sigma_1^{(0,1)} \qquad (24)$$

$$\Sigma_3^{(0,3)}/\Sigma_3^{(-1,2)} \cong \Sigma_1^{(0,1)} \otimes \Sigma_1^{(0,1)} \otimes \Sigma_1^{(0,1)} \qquad (25)$$

In the case general, the sl $_n$ -kdv $\mapsto$ burgers mapping can be seen as follows:

$$\Sigma_n^{(0,n)}/\Sigma_n^{(-1,n-1)} \cong \underbrace{\Sigma_1^{(0,1)} \otimes \Sigma_1^{(0,1)} \otimes \ldots \otimes \Sigma_1^{(0,1)}}_{n\ \ factors} \qquad (26)$$

In fact, this mapping presents a physical phenomenon binding a level where we have a high magnetic field ($B \sim \theta^{-1}$) but high degree of degeneration with a level which has a weak magnetic field $B$ and without degeneration. This splitting is called the *Zeeman Effect* in The Framework of Moyal Noncommutativity . for more detailed seeing[11].

## 3. MOYAL NONCOMMUTATIVE HIERARCHY EQUATIONS

In this section we will study the Lax representations of the NC Burgers and NC KdV equations with the higher-dimensional time evolution by the Lax-pair generating technique:

$$[\mathcal{L}, T_{mth-n} + \partial_{t_m}]_\star = 0, \qquad (27)$$

where the dimensions are given by $[t_m] = -m, [T_{mth-n}] = m$. The Lax representations (27) is derived from the compatible conditions of the NC linear systems:

$$L \star \psi = \lambda\psi,$$
$$\frac{\partial\psi}{\partial t_m} + T_{mth-n} \star \psi = 0. \qquad (28)$$

This time, Eq. (29) is not an evolution equation. However as the previous discussion, some geometrical meaning would be expected. Then, the existence of infinite number of hierarchy equations would suggest infinite-dimensional hidden symmetry which is expected to be deformed symmetry from commutative one.

### 3.1 Moyal NC Burgers hierarchy equation

Now let us take the other ansatz for the operator

$$L_{Burgers} = p + u_1$$
$$T_{(n)th-n} = p^n \star L_{Burgers} + T'_{(n)th-n}. \qquad (29)$$

Then the unknown part is reduced to $T'_{(n)th-n}$ which is determined so that Eq. (27) is a differential equation. The results are as follows.

- For n=1, the NC Lax equation gives the (*second-order*) NC Burgers equation. In fact the Lax pair in the noncommutative Moyal momentum formalism are explicitly given by

$$\begin{cases} L_{Burgers} = p + u_1(x,t) \\ T_{Burgers} = p^2 + 2u_1(x,t)p + u_1^2(x,t) + u_1'(x,t) \end{cases} \qquad (30)$$

gives *second-order* of NC Burgers equation

$$\frac{\dot{u}_1}{2\theta} + 2(1-\alpha)u_1 u_1' - \beta u_1'' = 0 \qquad (31)$$

With $\dot{u}_1 = \partial_{t_2} u_1 = \frac{\partial u_1}{\partial t_2}$ and $\lfloor \partial_{t_2} \rfloor = 2 = n + \lfloor L_{Burgers} \rfloor$. The equation (32) is linear for $\alpha = 1.$

For n=2, the NC Lax equation gives the *third-order* NC Burgers equation. In fact, the Lax pair is given by

$$\begin{cases} L_{Burgers} = p + u_1(t, x) \\ T_{3rd-h} = p^2 \star L_{Burgers} + T'_{3rd-h}, \end{cases} \qquad (32)$$

gives *third-order* of NC Burgers equation

$$\frac{\dot{u}_1}{2\theta} + (6\theta - b)(u_1 u_1')' - a u_1^{(3)} + (1 - c)(u_1^3)' = 0, \qquad (33)$$

The linearizable condition leads to the restricted situation $b = 6\theta,$ c=1 and a is an arbitrary real number, where the third-order Moyal NC Burgers equation (33) becomes trivial

$$\frac{\dot{u}_1}{2\theta} - a u_1^{(3)} = 0, \qquad (34)$$

whose associated Lax pair is

$$\begin{cases} L_{Burgers} = p + u_1 \\ T_{3rd-h} = p^3 + 3u_1 p^2 + 3u_1^2 p + (a + \theta^2)u_1'' + u_1^3. \end{cases} \qquad (35)$$

This result translates the strong point of the formalism Moyal momentum, where the conditions of linearizability of the Moyal NC Burgers equation (34), obtained from the 3rd stucture hierarchy, ( $b = 6\theta,$ c=1 and a is an arbitrary real number) is Large then that obtained in [8] where ( a=0, b=1, c=2 , d=1) for witch the NC Burgers equation is $\dot{u}_1 = 0.$

For n=3, the NC Lax equation gives the *fourth-order* NC Burgers equation. In fact, the Lax pair is given by

$$\begin{cases} L_{Burgers} = p + u_1(t, x) \\ T_{4th-h} = p^3 \star L_{Burgers} + T'_{4th-h}, \end{cases} \qquad (36)$$

gives *fourth-order* of NC Burgers equation

$$\begin{aligned} &(40\theta^2 - c - 2e)(u_1' u_1'')' + 4(1 - a)(u_1' u_1^3)' + (15\theta^2 - c)(u_1''' u_1)' \\ &+ (\theta^3 - d)u_1^{(4)} - (12 + b)(u_1'' u_1^2)' - 2(12 + b)(u_1' u_1' u_1)' + \frac{\dot{u}_1}{2\theta} = 0, \end{aligned} \qquad (37)$$

The linearizable condition leads to the restricted situation $a = 1, b = -12, \quad c = 15\theta^2, e = \frac{25}{2}\theta^2$ and d is an arbitrary real number, where the third-order Moyal NC Burgers equation (38) becomes trivial

$$(\theta^3 - d)u^{(4)} + \frac{\dot{u}}{2\theta} = 0, \qquad (38)$$

In this way, we can generate the higher-order Moyal NC Burgers equations. The ansatz for the (n+1)-th order is more explicitly given by

$$\begin{aligned} T_{(n+1)th-h} &= p^n \star L + T'_{(n+1)th-h} \\ &= p^{n+1} + \sum_{s=0}^{n} \theta^s c_n^s u^{(s)} p^{n-s} + \sum_{l=0}^{n} A_l p^{n-l}, \end{aligned} \qquad (39)$$

where $A_l$ are homogeneous polynomials of $u, u', u''$ and so on, whose degrees are $[A_{l+1}] = l + 1$.

### 3.2 Moyal NC KdV hierarchy equations

We consider the $sl_2$ -KdV operator $L_{KdV} = p^2 + u_2$ and the following ansatz for the operator

$$T_{(n+2)th-h} = p^n \star L_{KdV} + T'_{(n+2)th-h}. \qquad (40)$$

Then the unknown part is reduced to $T'_{(n+1)th-h}$ which is determined so that Eq. (27) is a differential equation. The results are as follows.

For n=1, the NC Lax equation gives the (*third-order*) NC KdV equation. In fact the Lax pair in the noncommutative Moyal momentum formalism are explicitly given by[8]

$$\begin{cases} L_{KdV} = p^2 + u_2(x, t) \\ T_{KdV} = p^3 + \frac{3}{2} p \star u_2(x, t) - \frac{3}{2}\theta u_2'(x, t) \end{cases} \qquad (41)$$

gives *third-order* of NC KdV equation

$$-\frac{\dot{u}_2}{2\theta} = \frac{3}{2} u_2 u_2' + \theta^2 u_2'''. \qquad (42)$$

with $\dot{u}_2 = \partial_{t_3} u_2,\ \lfloor \partial_{t_3} \rfloor = 1 + [L_{KdV}]$

For n=2, the NC Lax equation gives the *fourth-order* NC KdV equation. In fact, the Lax pair is given by

(43)

$$L_{KdV} = p^2 + u_2(t,x)$$
$$T_{4th} = p^2 \star L_{KdV} + T'_{4th},$$

gives *fourth-order* of NC KdV equation

$$\dot{u}_2 = \partial_{t_4} u_2 = 0, \tag{44}$$

For n=3, the NC Lax equation gives the *fifth-order* NC KdV equation. In fact, the Lax pair is given by

$$\begin{aligned} L_{KdV} &= p^2 + u_2(t,x) \\ T_{5th} &= p^3 \star L_{KdV} + T'_{5th}, \end{aligned} \tag{45}$$

gives *fifth-order* of NC KdV equation

$$\frac{\dot{u}_2}{2\theta} = \theta^4 u_2^{(5)} + \frac{5}{2}\theta u_2 u_2^{(3)} + 5\theta u_2' u_2'' + \frac{15}{8} u_2' u_2^2 = 0, \tag{46}$$

For n=4, the NC Lax equation gives the *sixth-order* NC KdV equation. In fact, the Lax pair is given by

$$\begin{aligned} L_{KdV} &= p^2 + u_2(t,x) \\ T_{6th} &= p^4 \star L_{KdV} + T'_{6th}, \end{aligned} \tag{47}$$

gives *sixth-order* of NC KdV equation

$$\dot{u}_2 = \partial_{t_6} u_2 = 0, \tag{48}$$

In this way, we can generate the higher-order Moyal NC KdV equations. The ansatz for the (n+2)-th order is more explicitly given by

$$\begin{aligned} T_{(n+2)th} &= p^n \star L + T'_{(n+2)th} \\ &= p^{n+2} + \sum_{s=0}^{n} \theta c_n^s u_2^{(s)} p^{n-s} + \sum_{l=0}^{n} A_{l+2} p^{n-l}, \end{aligned} \tag{49}$$

where $A_l$ are homogeneous polynomials of $u, u', u''$ and so on, whose degrees are $[A_{l+2}] = l+2$. From these calculate we will generalize these results, indeed,

For the case n=2k yields to the (2k+2)th-order of NC KdV equation

$$\dot{u}_2 = \partial_{t_{2k+2}} u_2 = 0, \tag{50}$$

the particular cases k=1 and k=2 correspond to the equations (45, 49).

For the case n=2k+1 yields to the (2k+3)th-order of NC KdV equations (43, 47)

$$\begin{aligned} n = 1 &: \quad \frac{\dot{u}_2}{2\theta} = \frac{3}{2} u_2 u_2' + \theta u_2''', \\ n = 3 &: \quad \frac{\dot{u}_2}{2\theta} = \theta^4 u_2^{(5)} + \frac{5}{2}\theta u_2 u_2^{(3)} + 5\theta u_2' u_2'' + \frac{15}{8} u_2' u_2^2 = 0, \\ &\dots \end{aligned} \tag{51}$$

with the transformation $\partial_{t_3} \to -2\theta\partial_{t_3}$ one can recover the following equations[6]

$$\begin{aligned} (u_2)_{t_1} &= u_2', \\ (u_2)_{t_3} &= \tfrac{3}{2} u_2 u_2' + \theta^2 u_2''', \\ (u_2)_{t_5} &= \tfrac{15}{8} u_2^2 u_2' + 5\theta^2 (u_2' u_2'' + \tfrac{1}{2} u_2 u_2''') + \theta^4 u_2^{(5)}, \\ (u_2)_{t_7} &= \tfrac{35}{16} u_2^3 u_2' + \tfrac{35}{8}\theta^2 (4u_2 u_2' u_2'' + u_2'^3 + u_2^2 u_2''') \\ &\quad + \tfrac{7}{2}(u_2 u_2^{(5)} + 3u_2' u_2^{(4)} + 5u_2'' u_2''')\theta^4 + \theta^6 u_2^{(7)}, \end{aligned} \tag{52}$$

...

## 4. NON STANDARD (PSEUDO-)BURGERS HIERARCHY

In this section we will see other situations for the number n (ie. 0, -1, -2, ...) in this case we have negative exponent p in the expression of the ansatz of T, and therefore in the second ansatz T'. We will propose a series of negative powers to p, this way of writing ansatz with negative powers will be call'd the Moyal pseudo-operators. Explicitly we have the following results

n = 0 case, the Lax equation

$$[L, T - \partial_t] = 0 \tag{53}$$

First ansatz

$$T = p^0 \circ L_1 + T' \tag{54}$$

where $[L_1]=[T]=[T_0]=1=[A_1]$, the Lax equation became

$$\partial_t u_1 = 2\theta\partial_x A_1 \tag{55}$$

and $[\partial_t]=[\partial_x]=1$; in this case the derivative with respect to time (dynamic) of $u_1$ coincides with the spatial derivative (static) of a field of same spin 1.

- n = -1 case, the Lax equation

$$\left[L, T + \partial_t\right]_\theta = 0 \quad (56)$$

First ansatz

$$T = p^{-1} * L_1 + T' \quad (57)$$

Here we have $[T] = [T'] = 0$ and

$$p^{-1} * L_1 = 1 + u_1 p^{-1} - \theta u_1^{'} p^{-2} + \theta^2 u_1^{''} p^{-3} - \theta^3 u_1^{(3)} p^{-4} + \ldots\ldots \quad (58)$$

Second ansatz

$$T' = \sum_{i=0}^{\infty} B_i * p^{-i} = \sum_{i=0}^{\infty} \tilde{B}_i p^{-i} \quad (59)$$

where $B_0 = \tilde{B}_0$ and $[B_i] = [\tilde{B}_i] = i$, the Lax equation became

$$\partial_{t_0} u_1 = 2\theta \partial_x B_0 \quad (60)$$

- n = -2 case, the associated Lax equation

$$\left[L, T + \partial_t\right]_\theta = 0 \quad (61)$$

First ansatz

$$T = p^{-2} * L_1 + T' \quad (62)$$

where $[T] = [T^{'}] = -1$ and

$$p^{-2} * L_1 = p^{-1} + u_1 p^{-2} - \theta u_1^{'} p^{-3} + \theta^2 u_1^{''} p^{-4} + \ldots\ldots \quad (63)$$

Second ansatz

$$T' = \sum_{i=0}^{\infty} C_{i-1} * p^{-i} = \sum_{i=0}^{\infty} \tilde{C}_{i-1} p^{-i} \quad (64)$$

where $C_{-1} = \tilde{C}_{-1}$ and $[C_i] = [\tilde{C}_i] = i$, the Lax equation became

$$\partial_{t_{-1}} u_1 = 2\theta \partial_x C_{-1} \quad (65)$$

- n = -3 case, the Lax equation

$$\left[L, T + \partial_t\right]_\theta = 0 \quad (66)$$

First ansatz

$$T = p^{-3} * L_1 + T' \quad (67)$$

where $[T] = [T'] = -2$ and

$$p^{-2} * L_1 = u_1 p^{-2} - u_1 p^{-3} - 3\theta^2 u_1^{'} p^{-4} + \ldots\ldots \quad (68)$$

Second ansatz

$$T' = \sum_{i=0}^{\infty} D_{i-2} * p^{-i} = \sum_{i=0}^{\infty} \tilde{D}_{i-2} p^{-i} \quad (69)$$

where $D_{-2} = \tilde{D}_{-2}$ and $[D_i] = [\tilde{D}_i] = i$, the Lax equation became

$$\partial_{t_{-2}} u_1 = 2\theta \partial_x D_{-2} \quad (70)$$

- n = -(k+1) case ($k \in IN$), we have the following equation

$$\partial_{t_{-k}} u_1 = 2\theta \partial_x E_{-k} \quad (71)$$

## 5. Conclusion

In this work, We used the Moyal Non commutative Lax generating technics, this study is seen as a generalization of our paper [8] for higher-order The KDV hierarchies. For this we have study Non commutative $sl_n$-KdV hierarchy equations by the means of Moyal Non commutative Lax generating technics. We have make some consistent assumptions shown to be essential in deriving the Lax pair of special noncommutative integrable systems, for this we calculate *second*, *third and fourth-orders* NC hierarchy equations for Burgers systems and we calculate the *third, fourth, fifth and sixth-orders* NC hierarchy equations for $sl_n$ -KdV system. We have presented also the conditions of having linear equations and giving the ansatz Process and the NC hierarchy equations in the general case.

## REFERENCES


[1]. Connes, M. R. Douglas and A. Schwarz, JHEP 9802 (1998) 003.

[2]. N. Seiberg and E. Witten, [arXiv:hep-th/9908142.]

[3]. M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, Phys. Lett. A 65 (1978) 185.

[4]. D. J. Korteweg and G. de Vries, Phil. Mag. 39 (1895) 422.

[5]. Das and Z. Popowicz, [arXiv:hep-th/0103063.]; A. Das and Z. Popowicz, [arXiv:hep-th/0104191].

[6]. Boulahoual and M. B. Sedra, , Chin. J. Phys. 43, 408 (2005).

[7]. Das, Integrable Models, World scientific, 1989.

[8]. O. Dafounansou, A. El Boukili and M. B. Sedra, Chin. J. Phys. 44, 274 (2006).

[9]. M. B. Sedra, Nucl. Phys. B 740 (2006) 243.

[10]. I.M. Gelfand and L.A. Dickey, J. Sov. Math.30 (1985) 1975; Funkt. Anal. Priloz.10 (1976) 13; 13(1979)13.

[11]. M. B. Sedra and A. El Boukili, Some Physical Aspects of Moyal Noncommutativity, Chin. J. Phys. Vol. 47 No. 3 2009