Wave-Front Curvature And Resolution Of The Abraham-Minkowski Controversy

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Abstract: This paper discusses about the momenta of photons in dielectric media. It is one of the unsolved problems in physics, especially optics. The problem took shape when, two scientists came up with two theories, mutually contradicting, with both of them supported by opposing experimental data. How can behaviour of light change in different experimental conditions? What have we missed? Shall we zoom in to the quantum level to explain this phenomenon? Or is it possible to explain it classically? I shall try to explain the variation of momentum with the help of curvature of the wave-front of incident light. We will discuss about the properties of curvature dependence of radiation pressure in media with higher refractive indices than vacuum. We will also derive a formula that accounts both for momentum and curvature, and also introduce the notion of 'Transition Constant'. It will explain the transition between kinetic and canonical momenta of photons in experiments.

Index Terms: Abraham-Minkowski Paradox; Curvature; Momentum; Radiation Pressure; Signum Function; Transition Constant; Wave-front.

1 INTRODUCTION

The motivation for this paper comes from the controversy between two scientists namely, Max Abraham and Hermann Minkowski. Both of their experiments on the momenta of light in a dielectric material suggested contradicting results. Firstly, the momentum of light according to the Einstein's mass-energy equivalence is given by,

$$p = \frac{E}{c}$$

According to Abraham's experiments, He put forward the fact that, as the light slows down by a factor of inverse of refractive index of the medium, so, the momentum must also decrease by the same factor. On the other hand, Minkowski's experiments suggested that, the momenta of photons indeed, did not decrease in the optical medium but increase by the factor of the refractive index. This depicted inconsistency of data. Firstly, it was said that the experimental errors had contributed to wrong results. The results were speculated to be unreliable. But soon, it was established that this was not the case. Both of them were correct. It was proposed that relativistic light had two momenta, one was the kinetic momentum and the other was canonical momentum. This meant that the vector E × H did not always signify the direction of electromagnetic power flow. Here, E is the electric field component and H is the magnetic field component of an electromagnetic wave. The same happens inside an optical medium, i.e. deviation from the 'poynting' vector. This means that there existed some fundamental anisotropy in dielectric media.

2 ABRAHAM'S EXPERIMENTATION

We have discussed about Abraham's observations in the previous section. He proposed that due to the decrease in the speed of light in the medium, there is a decrease in the momentum of photons.

All of his experimental results pointed towards an inverse dependence of momentum on refractive index. Thus, he formulated this fact as,

$$p = \frac{1}{n} \cdot \frac{E}{c} = \frac{\hbar\omega}{nc} = \frac{h\nu}{nc}$$

Where, p: momentum
n: refractive index
E: energy
c: speed of light
$\hbar$: reduced Planck's constant (h/2π)
ω: angular frequency
v: frequency

3 MINKOWSKI'S EXPERIMENTATION

Contrary to expectation, Minkowski suggested that the momentum of light in a medium is directly proportional to the refractive index. This happens even if light slows down. He formulated his results as,

$$p = \frac{n \cdot E}{c} = \frac{n\hbar\omega}{c} = \frac{nh\nu}{c}$$

Cleary, something different happens in this case. Abraham's momentum corresponds to kinetic momentum, and Minkowski's version corresponds to canonical momentum. This result depends upon the type of beam used in experimentation. The focussedness of the beam somehow determined the outcome of the experiment. This variation is discussed in the next section.

4 EXPERIMENTAL VARIATION

It was coherently said by researchers in May 2015, that,

> *"We illuminate a liquid…with an unfocused continuous-wave laser beam; we have observed a (reflected-light) focusing effect…in quantitative agreement with Abraham momentum."*

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"We focused the incident beam tightly … we observed a de-focusing reflection…in agreement with Minkowski momentum transfer."

This is the key point in deriving a unified equation between both relations. In both the cases, the difference lies in the fact whether the beam of light is focused or not. As per the wave-particle duality of the electromagnetic radiation, especially light, the property that differentiates between an unfocussed and a focused beam of light is the curvature of the wave-front. (See Fig. 1) Taking in consideration the sign conventions of ray optics, we can define the curvature of the wave-front to be positive or negative. Now, let us build some concepts in the next section.

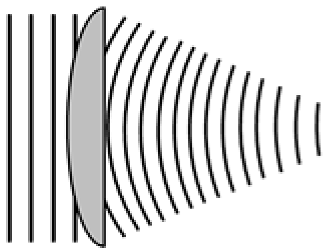

Fig. 1: *Conversion of plane wave-front of light into spherical wave-front by a convex lens.*

5 BUILDING THE CONCEPT

When we deal with lenses, they are the optical instruments that change plane wave-fronts to spherical wave fronts. So, we will have to construct an expression that accounts for the scalar curvature of the sphere. So, the scalar curvature for any surface (See Fig. 2, Next page) in Euclidean space is as follows;

$$S = \frac{2}{\rho_1 \rho_2}$$

Where, ρ1 and ρ2 are the principle radii of the surface at the point.

$$S = \frac{2}{R^2}$$

So for a 2-sphere, both the principal radii are equal. Let them be R. So, we get the scalar curvature of a 2-sphere as,

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{2f} = \frac{(n-1)}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{R} = \frac{(n-1)}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{R^2} = \frac{(n-1)^2}{4}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)^2$$

$$S = \frac{2}{R^2} = \frac{(n-1)^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)^2$$

Let us assume that, the radius of curvature of the wave-front is equal to the resultant curvature of the lens using which the beams are concentrated. Now, we use the lens maker's formula, to setup a relation between curvature of wave front and the radii of curvature of both the surfaces of the lens taken into consideration. Let, f be the resultant focal length of the lens, n be its refractive index, R1 and R2 be the radii of curvature of the lens.

$$p = n^{f(S)} \cdot \frac{\hbar\omega}{c}$$

Now, we can notice in both Abraham's and Minkowski's version of radiation pressure, the variation occurs in the dependence of momentum on refractive index. In one case, it is directly proportional, and in the other, it's inversely proportional. So, in the expression we derive, we require that the power of the refractive index should vary from -1 to +1 being a function of the scalar curvature of the wave-front. So, f (s) is some function of the scalar curvature S, of the wave-front. So, now we need to find this function. Subsequently, it is intuitive that, this function must be bounded and has values only in the interval [-1, 1]. It should also have positive definite values for all numbers greater than a particular value of transition, and negative definite values for all numbers lesser than the same transition value. This transition value corresponds to the transition between the direct and inverse dependence of momentum on refractive index. So we also need to find this transition curvature value. Let us recall, when an unfocused beam of light is used in the experiment, we find numerical agreement with Abraham momentum. So an unfocused light has a plane wave-front, which corresponds to zero curvature. Thus, an unfocussed beam or a divergent beam (negative wave-front curvature), leads to inverse dependence on refractive index. And a focussed beam, having positive wave-front curvature, leads to direct dependence of refractive index. Consequently, the function should have only the value +1 for positive curvatures and -1 for 0 and negative curvatures. The function which satisfies such conditions is the Signum function or the Sign Function. Let's now derive the equation.

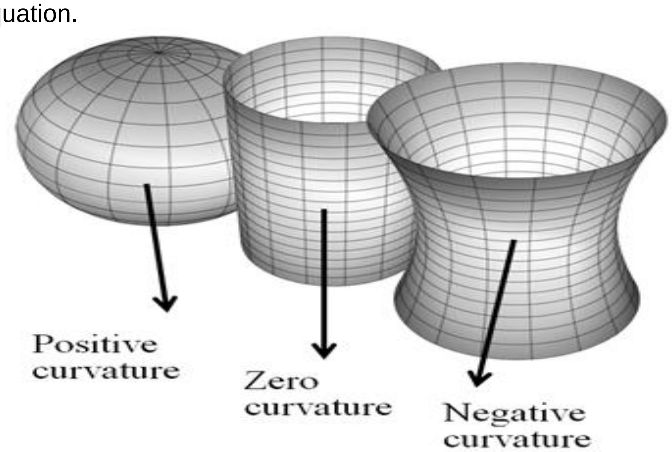

Fig. 2: *A figure showing negative, zero and positive curvature conventions in three dimensional manifolds.*

5 DERIVATION

Consider a lens (may be convex or concave), which manipulates the curvature of the plane wave-front of light. Let the radii of curvature of the lens be 'R1' and 'R2'. Let, the refractive index of the lens be 'μ'. And that of the material in which experiment is performed be 'n'. As per our parameters, the wave-front curvature is given by,

$$S = \frac{(\mu-1)^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)^2$$

Whenever wave-fronts converge or diverge, they do so in the form of concentric spheres. We know that curvature is proportional to square of distance from the optical centre of the lens, to any instantaneous converging wave-front (positive curvature) during propagation, as the principal radii tend to zero. This distance dependence is important, as it gives us the information about at what point in space does the wave-front collide with the medium interface. Because, there will be a difference in wave-front curvature if the lens is placed close or far away from the medium. (See Fig. 3, Next Page) This will also help to maintain dimensional consistency in the expression we need to derive. Let the distance between the lens and the medium interface be 'd'. So we define γ, i.e. the distance dependent curvature as,

$$\gamma = d^2\frac{(\mu-1)^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)^2 = d^2 \cdot S$$
$$where, 0 < d \leq f$$

Y is dimensionless.

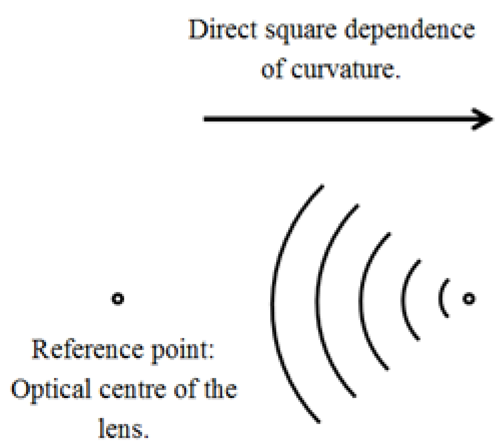

Fig. 3: *Figure showing direct square dependence of curvature on distance. (It must not be confused with inverse square dependence of curvature on principal radii.)*

Now, Let us come back to the Signum function. We will make γ as the argument of the signum function, so that we get values, -1 and +1 based on wave-front curvature. The signum of γ must be the power of n, the refractive index of medium. Thus we come at an expression,

$$p = n^{sgn(\gamma)} \cdot \frac{\hbar\omega}{c}$$

Where,

$$sgn(x) = \begin{cases} -1 & if\ x < 0 \\ 0 & if\ x = 0 \\ 1 & if\ x > 0 \end{cases}$$

& $$\gamma = d^2\frac{(\mu-1)^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)^2$$

There are some problems with this expression. Let us discuss and rectify them in the next section.

6 A BETTER APPROXIMATION

We all know that the Signum function is not continuous at x = 0. Nature won't take any jumps! Or the light finds zero curvature and as soon it crosses it, there is radical change in the momentum of light! This of course does not happen. So, using the Signum function will only describe the, behaviour of nature at all curvatures except zero. This is not desirable. So we need to use a continuous approximation of the Signum function. This can be accomplished by using the Hyperbolic Tangent function, which asymptotically approaches the Signum function under certain conditions. This will now completely describe the characteristic momenta of photons in media. So, we have,

$$sgn(x) \approx \tanh(kx) \quad where,\ k \gg 1$$

So, we replace sgn(x) by tanh(kx) where, k is a constant and k≫1. Thus by replacing Signum function by tangent function we get momentum to be,

$$p = n^{\tanh(\eta_0\gamma)} \cdot \frac{\hbar\omega}{c}$$

Now, I've introduced a new constant here, in this context. It is η0, which is 'Transition Constant'. This decides how smooth transition occurs between direct and inverse dependence of momentum on refractive index. This is universal to the behaviour of light and can be determined experimentally. This expression can successfully and accurately predict momentum of photons in medium dependent on signed wave-front scalar curvature. Let's discuss the features and properties of the 'Transition Constant' in the next section.

7 THE TRANSITION CONSTANT

As we know that, the transition constant controls how transition occurs between direct and inverse dependence of momentum on refractive index, it can be determined experimentally by obtaining measured momentum in the neighbourhood of transition curvature that is zero curvature. The transition constant in fact, encodes the exceptional behaviour of light quanta in dielectric media in the transition period. Once we know those momentum values corresponding to particular curvatures near null curvature then we can calculate η0 by the formula as on the next page,

$$\eta_0 = \frac{1}{\gamma}\tanh^{-1}\left(\frac{\log\left(\frac{pc}{\hbar\omega}\right)}{\log n}\right)$$

This constant is very fundamental to the behaviour of light. It does not depend on the medium, frame of reference, wavelength of light or its energy etc. The determination of this constant will demand accurate experiments. Let us graphically understand the significance of the transition constant. (See Fig.4).

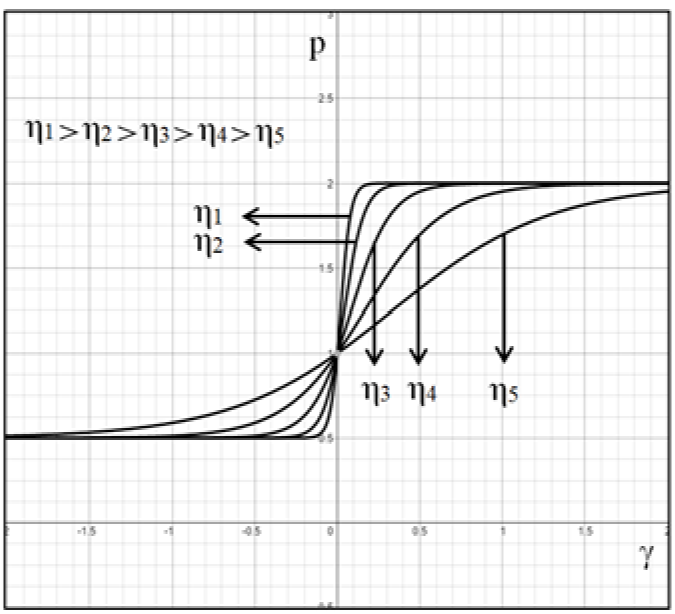

Fig. 4: *A plot of momentum versus wave-front curvature at different transition constants. Greater is the transition constant, steeper is the transition. And when η0 → ∞ , tanh(x) → sgn(x).*

8 ANALYSIS

Let us discuss some of the plots of the obtained formula. (See Fig. 5) It is the graph that describes the variation of momentum with curvature of wave-front. For positive curvature, which is a convergent beam, the momentum is directly proportional to refractive index and for negative curvature; momentum is inversely proportional to refractive index. And, in between there is a smooth transition between direct and inverse dependence. The smoothness of transition depends on value of η0. It displays momenta of photons having different frequencies. We can easily interpret that higher is the frequency, higher is the momentum, for both, negative and positive curvature. The y-intercept is always equal to the momentum of plane wave-front i.e. zero curvature. Now (See Fig.6, Next Page). It shows different transitions of momenta for different media varying in refractive indices. As, Minkowski momentum is directly proportional to refractive index, so higher the refractive index, higher the momentum, which is depicted in the positive curvature zone of the graph. The opposite is illustrated in the negative curvature zone; Abraham momentum tends to zero, as refractive index tends to infinity. Consequently, the derived formula proves to be extremely consistent with the physical world and its behaviour.

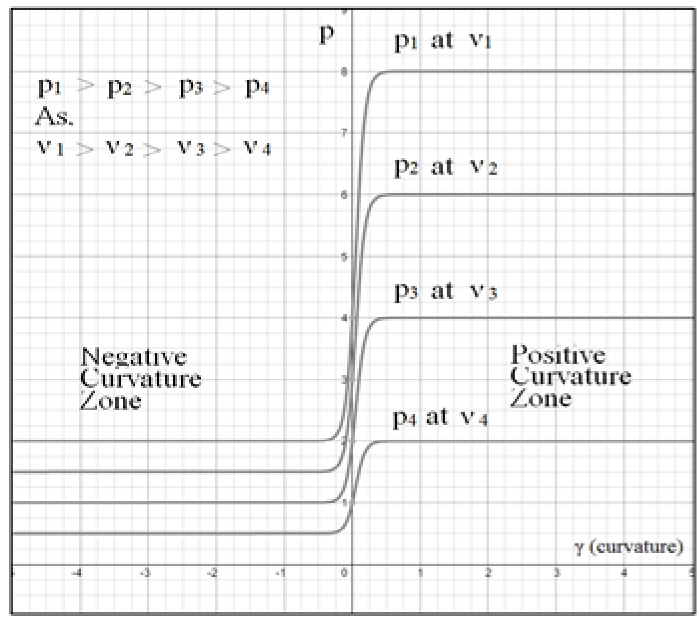

Fig. 5: *This plot depicts different momentum transitions for different frequencies of light. Thus momentum is directly proportional to frequency.*

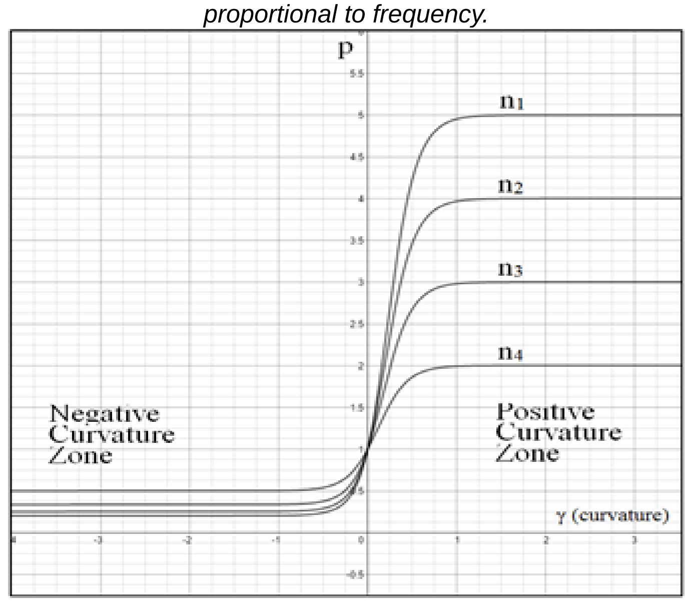

Fig. 6: *This plot depicts different momentum transitions in media having different refractive indices. Higher the refractive index higher the Minkowski momentum but lower the Abraham momentum.*

9 CONCLUSION

The formula hence derived predicts a smooth transition from direct to inverse dependence of momentum on refractive indices based on wave-front curvature. The smoothness is governed by the put forward transition constant. The obtained expression explains both Minkowski and Abraham momentum and unifies them. It gives mathematical explanation of behaviour of light beams having a defined curvature and its curvature dependent interaction with the dielectric. This is simply another manifestation of the kinetic momentum and the canonical momentum. This has indeed unified the observation of two scientists, and will help us understand the fundamental properties of light to the fullest. This is not just a solution to a paradox, but has deep meanings that will help us to question ourselves on the exactness of our knowledge and invigorate our curiosity. Hope for a better tomorrow!

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