Uniserial Dimension Of Module $\mathbb{Z}_m \times \mathbb{Z}_n$ Over $\mathbb{Z}$ Using Python

Arifin S., Garminia H.

Abstract: Recently, the notion of the uniserial dimension of a module over a commutative ring that measures of how far the module deviates from being uniserial was introduced by Nazemian in 2014. In this article, we give some methods to determine the uniserial dimension of a finitely generated primary module over a principal ideal domain, especially $u.s.\dim(\mathbb{Z}_m \times \mathbb{Z}_n)_{\mathbb{Z}}, m, n \in \mathbb{Z}$ using Python. It is well known that finitely generated primary modules over a principal ideal domain can be decomposed as a direct sum of finite cyclic submodules where the orders of the cyclic generators are called the elementary divisors of the module. We show that $u.s.\dim(\mathbb{Z}_m \times \mathbb{Z}_n)_{\mathbb{Z}}$ is depends on prime factorization of m and n.

Index Terms: Uniserial dimension, module $\mathbb{Z}_m \times \mathbb{Z}_n$ over $\mathbb{Z}$, Python

1 INTRODUCTION

Uniserial modules are a generalization of valuation rings. It well known that a valuation ring can be identified through a collection of all ideals that are ordered by inclusion. The notion of valuation rings was studied in [6], [7], [11] and [13]. On the other hand, uniserial modules can be identified through a collection of all submodules that is totally ordered by inclusion. The properties of a uniserial module have been studied by [5], [4], [9], and [20]. A measure of how far a module deviates from being uniserial is called uniserial dimension. The notion of uniserial dimension was introduced by Nazemian [14].

Some properties of modules with uniserial dimension have been studied. Nazemian [14] showed that for any ring R and ordinal number α, there exists an R-module with uniserial dimension α. Furthermore, a commutative ring R is Noetherian if and only if any finitely generated R-module has uniserial dimension. A commutative ring R is Noetherian if and only if every finitely generated R-module has uniserial dimension. Moreover, A commutative ring R is Artinian if and only if every finitely generated R-module has finite uniserial dimension. If an R-module M has a uniserial dimension, then every factor module of M has a uniserial dimension. In the literature, Nazemian [14] also showed that every Noetherian module has a uniserial dimension. Furthermore, every Artinian module, which also Noetherian, has a finite uniserial dimension.

Roman [15] showed that finitely generated primary modules over a principal ideal domain can be decomposed as a direct sum of finite cyclic submodules where the orders of the cyclic generators are called the elementary divisors of the module. The first step in the decomposition of a finitely generated module M over a principal ideal domain R is $M = M_{free} \oplus M_{tor}$ where M_{free} is a finitely generated free R-module and M_{tor} is the finitely generated torsion R-module. Next, we turn our attention to the decomposition of the torsion part M_{tor}, i.e. to decompose M_{tor} into a direct sum of primary submodules. After that, the primary module part can be decomposed into a direct sum of cyclic submodules (aslo see [1], [3], [12] and [16]).

The above facts lead to a question concerning a method to determine the uniserial dimension of the torsion part, especially $M = (\mathbb{Z}_m \times \mathbb{Z}_n)_{\mathbb{Z}}$, i.e. $u.s.\dim(\mathbb{Z}_m \times \mathbb{Z}_n)_{\mathbb{Z}}$ using Python, and it is the question beeing adressed by this paper. We begin with a review of modules with finite uniserial dimension in Session 2. The main result, the methods and a Python's code to determine $u.s.\dim(\mathbb{Z}_m \times \mathbb{Z}_n)_{\mathbb{Z}}$ in Session 3. Examples and discussion about the output of program is in Session 4.

2 REVIEW OF MODULES WITH FINITE UNISERIAL DIMENSION

Through out this paper, we assume all rings are commutative with unity. The development of the notion of uniserial dimension of a module can not be separated from the notion of submodules and their quotient modules (see [14]). This notion is related to ordinal numbers which is developed by transfinite induction (see Stoll [19]). For each ordinal number $\alpha \geq 1$ and a ring R, a collection of R-modules ζ_α can be generated by transfinite induction, starting from ζ_1, using the collection of R-modules ζ_β for all ordinal number $\beta < \alpha$. Especially, ζ_1 contains all non-zero uniserial R-modules.

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Definition 2.1 ***Nazemian [14].*** *For any ordinal number $\alpha \geq 1$ and R-module M, we define:*

1. $\zeta_1 = \{M \mid M \text{ uniserial } R-\text{modules}\}$
2. $\zeta_\alpha = \{M \mid M\ R-\text{module}, (\forall N < M)$ $\left(M/N \not\cong M \Rightarrow M/N \in \bigcup_{\beta<\alpha} \zeta_\beta\right)\}$

Referring to the above definition, the collection ζ_2 contains all R-modules M that are not uniserial and for every submodule N of M, if $M/N \not\cong M$ implies M/N is uniserial. A module M is said to have a uniserial dimension if there are ordinal numbers α such that $M \in \zeta_\alpha$. It is easy to show that if $M \in \zeta_\beta$ and $\beta < \alpha$ then $M \in \zeta_\alpha$. Furthermore, a module M is said to have a uniserial dimension α, denoted $u.s.dim(M) = \alpha$, if α is the smallest ordinal number such that $M \in \zeta_\alpha$. This notion is stated in the following definition.

Definition 2.2 ***Nazemian [14].*** *Let R be a ring and M be an R-module.*

1. For $M \in \zeta_\alpha$, the minimal ordinal number α is said universal dimension from M, denote $u.s.dim(M) = \alpha$.
2. For $M = 0$, $u.s.dim(M) = 0$
3. For $M \neq 0$ but $M \notin \zeta_\alpha$ for any ordinal number $\alpha \geq 1$, we said that "$u.s.dim(M)$ is not defined" or "M has no uniserial dimension"

Let $M = \mathbb{Z}_5$ be a module over $\mathbb{Z}$, then $u.s.dim(M) = 1$ since M is uniserial, and for general, if $M = \mathbb{Z}_p$ where $p \in \mathbb{Z}$ is prime, is a module over $\mathbb{Z}$, then $u.s.dim(M) = 1$ (see Lemma 3.3). The following is the module M with $u.s.dim(M) = 2$ (see Lemma 3.5).

Example 2.3

Let $M = \mathbb{Z}_7 \oplus \mathbb{Z}_7$ be a module over $\mathbb{Z}$, we will show that $u.s.dim(M) = 2$. Note that M is generated by $\{(1,0),(0,1)\}$. Consider that all proper submodules and their factor modules of M are:

1) $N_1 = [[0, 0], [0, 1], [0, 2], [0, 3], [0, 4], [0, 5], [0, 6]]$ such that $M/N_1 \cong \mathbb{Z}_7 \oplus \{0\} \cong \mathbb{Z}_7 \in \zeta_1$,

2) $N_2 = [[0, 0], [1, 0], [2, 0], [3, 0], [4, 0], [5, 0], [6, 0]]$ such that $M/N_2 \cong \{0\} \oplus \mathbb{Z}_7 \cong \mathbb{Z}_7 \in \zeta_1$,

3) $N_3 = [[0, 0], [1, 1], [2, 2], [3, 3], [4, 4], [5, 5], [6, 6]]$ such that $M/N_3 \cong \{0\} \oplus \mathbb{Z}_7 \cong \mathbb{Z}_7 \in \zeta_1$,

4) $N_4 = [[0, 0], [1, 2], [2, 4], [3, 6], [4, 1], [5, 3], [6, 5]]$ such that $M/N_4 \cong \{0\} \oplus \mathbb{Z}_7 \cong \mathbb{Z}_7 \in \zeta_1$,

5) $N_5 = [[0, 0], [1, 3], [2, 6], [3, 2], [4, 5], [5, 1], [6, 4]]$ such that $M/N_5 \cong \{0\} \oplus \mathbb{Z}_7 \cong \mathbb{Z}_7 \in \zeta_1$,

6) $N_6 = [[0, 0], [1, 4], [2, 1], [3, 5], [4, 2], [5, 6], [6, 3]]$ such that $M/N_6 \cong \{0\} \oplus \mathbb{Z}_7 \cong \mathbb{Z}_7 \in \zeta_1$.

7) $N_7 = [[0, 0], [1, 5], [2, 3], [3, 1], [4, 6], [5, 4], [6, 2]]$ such that $M/N_7 \cong \{0\} \oplus \mathbb{Z}_7 \cong \mathbb{Z}_7 \in \zeta_1$.

8) $N_8 = [[0, 0], [1, 6], [2, 5], [3, 4], [4, 3], [5, 2], [6, 1]]$ such that $M/N_8 \cong \{0\} \oplus \mathbb{Z}_7 \cong \mathbb{Z}_7 \in \zeta_1$.

Therefore, for any N submodules of M, $M/N \in \zeta_1$, so we get $M \in \zeta_2$. On the other hand, there exist submodules N_1, N_2 of M such that N_1 Ú N_2 and N_1 Û N_2 which means M is not uniserial, so we get $M \notin \zeta_1$. This implies that $u.s.dim(M) = 2$.+

Some properties of modules that has a uniserial dimension already found (see [14]). An R-module M has a uniserial dimension if and only if for every ascending chain $M_1 \leq M_2 \leq \ldots$ of submodules of M, there exists $n \geq 1$ such that M/M_n is uniserial or $M/M_n \cong M/M_k$ for all $k \geq n$. Therefore, every Noetherian module has a uniserial dimension. Moreover, a commutative ring R is Noetherian if and only if every finitely generated R-module has a uniserial dimension. If M is a module of finite length, then M has a uniserial dimension. Therefore, a commutative ring R is Artinian if and only if every finitely generated R-module has finite uniserial dimension.

The uniserial dimension of a regular module over a principal ideal domain has been shown in [18], i.e., if R is a principal ideal domain and $M = (R)_R$, then $u.s.dim(M) = 1$ or $u.s.dim(M) = \infty$. Therefore, for module $M = (\mathbb{Z})_\mathbb{Z}$, $u.s.dim(M) = \infty$. On the other hand, In [17], a method to determine the uniserial dimension of module $M = (\mathbb{Z}_n)_\mathbb{Z}$ using Python has been studied. From this result and the literature, there is no tools to determine the uniserial dimension of module $M = (\mathbb{Z}_m \times \mathbb{Z}_n)_\mathbb{Z}, m, n \in \mathbf{Z}^+$ and will be discussed in the next section using Python.

3 Methods

In this section, we will study the methods to counting the uniserial dimension of the module $M = (\mathbb{Z}_m \times \mathbb{Z}_n)_\mathbb{Z}$, i.e. $u.s.\dim(\mathbb{Z}_m \times \mathbb{Z}_n)_\mathbb{Z}$, $m, n \in \mathbb{Z}^+$ using Python. We will see this method in Corollary 3.7., which is the main result of this section.

We will need the following lemma first to get all cyclic submodules of $(\mathbb{Z}_n)_\mathbb{Z}$, that is the set generated by all factors of n.

Lemma 3.1 ***Gallian [8]***. *For each positive divisor k of n, the set $\left\langle \frac{n}{k} \right\rangle$ is the unique subgroup of $\mathbb{Z}_n$ of order k. Morover, these are the only subgroups of $\mathbb{Z}_n$.*

Lemma 3.1. ***Huang [10]***. *The following statements are true:*

1. Let m and n be positive integers. If $\gcd(m, n) = 1$ (i.e. m and n are relative prime), then $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$, and $(1,1)$ is a generator of $\mathbb{Z}_m \times \mathbb{Z}_n$.
2. The group $\prod_{i=1}^{n} \mathbb{Z}_{m_i}$ is cyclic and is isomorphic to $\mathbb{Z}_{m_1 \cdots m_n}$ if and only if the numbers m_i for $i = 1, \ldots, n$ are pairwise relative prime, that is, the gcd of any two of them is 1.
3. If a positive integer n is factorized as a product of powers of distinct prime numbers: $n = p_1^{n_1} \ldots p_r^{n_r}$ then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \times \ldots \times \mathbb{Z}_{p_r^{n_r}}$.

Lemma 3.2 is a powerful tool to prove Theorem 3.6. This lemma says that if a module has a uniserial dimension α, then for any $0 \le \beta \le \alpha$ there exists a factor module with a uniserial dimension β.

Lemma 3.2. ***Nazemian [14]***. *If M is an R-module and $u.s.dim(M) = \alpha$, then for any $0 \le \beta \le \alpha$, there exists a factor module M/N of M such that $u.s.dim(M/N) = \beta$.*

Since $M = (\mathbb{Z}_n)_\mathbb{Z}$ with $n = p^n$, p is prime and $n \in \mathbb{Z}^+$, is uniserial, then we will get the following lemma.

Lemma 3.3. Samsul [17]. *Let $\mathbb{Z}_n$ be the integer modulo n with $n = p^n$ and p are prime and $n \in \mathbb{Z}^+$. Then $u.s.dim(\mathbb{Z}_n)_\mathbb{Z} = 1$.*

In general, the following results will be obtained.

Theorem 3.4. Samsul [17]. *Let $\mathbb{Z}_n$ be the integer modulo n with $n = p_1 \ldots p_k$ and p_i are distinct prime, $i = 1, \ldots, k$. Then $u.s.dim(\mathbb{Z}_n)_\mathbb{Z} = k$.*

Proof. Let $M = \mathbb{Z}_n$ be a module over $\mathbb{Z}_n$ with $n = p_1 \ldots p_k$ are distinct primes. The proof is by induction on k. The case $k = 1$ is clear by Lemma 3.3. Now let $k = l$, then $n_l = p_1 \ldots p_l$. Therefore by hypothesis, $u.s.dim(\mathbb{Z}_{n_l}) = l$. It means $\mathbb{Z}_{n_l} \in \xi_l$ and $\mathbb{Z}_{n_l} \notin \xi_{l-1}$. We have to show that for $k = l + 1$ the statement is true. Let $n_{l+1} = p_1 \ldots p_{l+1} = (p_1 \ldots p_l)\, p_{l+1}$. Let N be any proper submodule of $M_{l+1} = \mathbb{Z}_{n_{l+1}}$ and $a \in N$, $a \neq 0$ where $a = p_1 \ldots p_{(l+1)-1} = p_1 \ldots p_l$. Without lost the generality, there exist prime element $\overline{p} \in \mathbb{Z}_{n_{l+1}}$, $gcd(a, p) = 1$ such that $b = pa = p(p_1 \ldots p_l)$. So, there exist a submodule $\langle b \rangle = pa$ of M_{l+1} such that $\langle b \rangle \subseteq N \subseteq M_{l+1}$ and $M_{l+1}/\langle b \rangle \cong M_l$. By hypothesis, $M_{l+1}/\langle b \rangle \in \xi_l$ and $M_{l+1}/\langle b \rangle \notin \xi_{l-1}$. Note that:

$$M_{l+1}/N \cong \left(M_{l+1}/\langle b \rangle\right) / \left(N/\langle b \rangle\right) = M_l / \left(N/\langle b \rangle\right).$$

By Lemma 2.4, $M_{l+1} \in \xi_{l+1}$ and $M_{l+1} \notin \xi_{(l+1)-1}$, so $u.s.dim(M_{l+1}) = l + 1$, which means that for $k = l + 1$ the statement is true. Now, we can say that $u.s.dim(\mathbb{Z}_n) = k$ for $n = p_1 \ldots p_k$, $k \in \mathbb{Z}^+$.+

Lemma 3.5. says that for a $\mathbb{Z}$-module $M = \mathbb{Z}_p \times \mathbb{Z}_p$, the uniserial dimension of M is two. Note that if $M = (\mathbb{Z}_p \times \mathbb{Z}_p)_\mathbb{Z}$, $p \in \mathbb{Z}^+$ are prime, and N is any proper submodule of M then N is cyclic. But this property is only true for this case, and will be used in the following lemma.

Lemma 3.5. *Let $M = \mathbb{Z}_p \times \mathbb{Z}_p$ be a module over $\mathbb{Z}$ and $p \in \mathbb{Z}^+$ are prime. Then $u.s.dim(M) = 2$.*

Proof. Let $M = (\mathbb{Z}_p \times \mathbb{Z}_p)_\mathbb{Z}$ where $\mathbb{Z}_p$ are simple. Let N be any proper submodule of M. Then there exists $b \in N$

such that $b=(0,\alpha)$ or $b=(\alpha,0)$ for some $\alpha\in\mathbf{Z}$. Therefore, there exist $\langle b\rangle$ submodule of M such that $\langle b\rangle\subseteq N\subseteq M$ and $M/\langle b\rangle\cong \mathrm{Z}_p\in\xi_1$. Note that $N=\langle b\rangle$, so we get $M/N\in\xi_1$. It means $M\in\zeta_2$. On the other hand, there exists $N_1=\{0\}\oplus \mathrm{Z}_p, N_2=\mathrm{Z}_p\oplus\{0\}$ submodules of M such that $N_1\nsubseteq N_2$ and $N_2\nsubseteq N_1$, so we get that M is not uniserial. It means $M\notin\zeta_1$. This implies that $u.s.dim(M)=2$ by Definition 2.1.+

From Lemma 3.5, we will get the following result, i.e., for a module $M=(\mathrm{Z}_m\times\mathrm{Z}_n)_{\mathbf{Z}}, m,n\in\mathbf{Z}^+$ where $m=p^k$, $n=p^l$, and $k\geq 1$, $l\geq 1$, then we get that the uniserial dimension of M is $k+l$.

Theorem 3.6. *Let* $M=\mathrm{Z}_m\times\mathrm{Z}_n$, $m,n\in\mathbf{Z}^+$ *be a module over* $\mathbf{Z}$ *where* $m=p^k$, $n=p^l$, *and* $k\geq 1$, $l\geq 1$. *Then* $u.s.dim(M)=k+l$.

Proof. The proof is by induction on $k+l$ with $k+l\geq 2$. The case $k+l=2$ which is equivalent to $k=1$ and $l=1$, is clear by Lemma 3.5. Suppose the lemma holds for all k,l with properties $k\geq 1, l\geq 1$ and $2\leq k+l\leq r$. Let $M=\mathrm{Z}_m\times\mathrm{Z}_n$ with $m=p^k$, $n=p^l$, and $k+l=r+1$. Let N be any proper submodule of M. Let $a\in N, a\neq 0$. Then $a=(\alpha_1,\alpha_2)$ for some $\alpha_1,\alpha_2\in\mathbf{Z}$ with $m\nmid\alpha_1$ or $n\nmid\alpha_2$. Let us consider two cases: $m\mid\alpha_1$ and $m\nmid\alpha_1$.

1) If $m\mid\alpha_1$ then $\alpha_1=mq=0$, $q\in\mathbf{Z}$, so $a=(0,\alpha_2)\neq 0$ with $n\nmid\alpha_2$. Let l_1 be the biggest non-negative integer such that $n_1\mid\alpha_2$ for $n_1=p^{l_1}$. There exists $c\in\mathbf{Z}$ such that $b=ca=c(0,\alpha_2)=(0,c\alpha_2)=(0,n_1)$. In this case we have $M/\langle a\rangle\cong\mathrm{Z}_m\oplus\langle n_1\rangle$ with $n_1=l_1<l$. Hence $k+l_1\leq k+l\leq m+1$, which implies that $M/\langle a\rangle\in\xi_{k+l_1}\subseteq\xi_m$. As a result, $M/N\cong(M/\langle a\rangle)/(N/\langle a\rangle)\in\xi_m$ by Lemma 3.2.
2) If $m\nmid\alpha_1$, then $\alpha_1=mq+t=0+t=t$, $q,t\in\mathbf{Z}$ so $a=(t,\alpha_2)\neq 0$. Let $c\in\mathbf{Z}$ such that $b=ca=c(t,\alpha_2)=(ct,c\alpha_2)=(m_1,c\alpha_2)$ with $m_1=p^{k_1}, k_1<k$. We obtain $M/\langle b\rangle$ is cyclic or $M/\langle b\rangle\cong\mathbf{Z}_{m_2}\oplus\mathbf{Z}_{n_1}$, where $m_2=p^{k_2}$, $n_1=p^{l_1}$, $k_2\leq k_1<k$ and $l_1\leq l$. Hence $k_1+l_1\leq k+l\leq m+1$, which implies $M/\langle b\rangle\in\xi_{k_1+l_1}\subseteq\xi_m$. As a result, $M/N\cong(M/\langle b\rangle)/(N/\langle b\rangle)\in\xi_m$ by Lemma 3.2.

Therefore, for $k+l=r+1$ the statement is true. This implies that $u.s.dim(M)=k+l$.+

From Lemma 3.1, Lemma 3.5, and Theorem 3.6, we will get the following corollary. This is the main result of how to counting the uniserial dimension of the module $M=(\mathrm{Z}_m\times\mathrm{Z}_n)_{\mathbf{Z}}, m,n\in\mathbf{Z}^+$.

Corollary 3.7. *Let M be a module over ring* $\mathbf{Z}$.

a) *If* $M=(\mathrm{Z}_p\times\mathrm{Z}_p)_{\mathbf{Z}}$ *then* $u.s.dim(M)=2$.
b) *If* $M=(\mathrm{Z}_m\times\mathrm{Z}_n)_{\mathbf{Z}}$ *for* $m=p^k, k\geq 1$ *and* $n=p^l, l\geq 1$, *then* $u.s.dim(M)=k+l$.
c) *If* $M=(\mathrm{Z}_m\times\mathrm{Z}_n)_{\mathbf{Z}}$ *for* $m=p_1\ldots p_k$, $n=q_1\ldots q_l$, *and* p_i,q_j *are distinct prime,* $i=1,\ldots,k$, $j=1,\ldots,l$, *then* $u.s.dim(M)=k+l$.

We will combine between Corollary 3.7. and prime factorization of the integer (see Aditya [2]) to make a program using Python 2.7.14 to determine the uniserial dimension of the module $M=(\mathrm{Z}_m\times\mathrm{Z}_n)_{\mathbf{Z}}$. The reader can copy and paste the code into Python, and press F5 to execute the program. On the other hand, to see the lattice of submodules of any modules $(\mathrm{Z}_n)_{\mathbf{Z}}$, we can use Wolfram CDF Player and download the file from their official website (see Wolfram [21]). The main program is as below.

```
print "============================================="
print "Uniserial Dimension of a Module Zm x Zn over Z"
print "---------------------------------------------"
x = input("Input m:")
y = input("Input n:")
print "We have a module M = Z_",x,"x Z_",y, "over ring
Z"

def faktorisasiprima(x):
    factorlist=[]
    loop=2
    while loop<=x:
        if x%loop==0:
            x/=loop
            factorlist.append(loop)
        else:
            loop+=1
    return factorlist

def faktorisasiprima(y):
    factorlist=[]
    loop=2
    while loop<=y:
        if y%loop==0:
            y/=loop
            factorlist.append(loop)
        else:
            loop+=1
```

```
    return factorlist
print     "Prime     factorization     of     ",x,":",
faktorisasiprima(x), "and",y,":", faktorisasiprima(y)

h1 = faktorisasiprima(x)
ph1 = set(h1)
h2 = faktorisasiprima(y)
ph2 = set(h2)
vdimku = len(h1)+len(h2)
print "Therefore, u.s.dim(M) =", vdimku
print "---------------------------------------------"
```

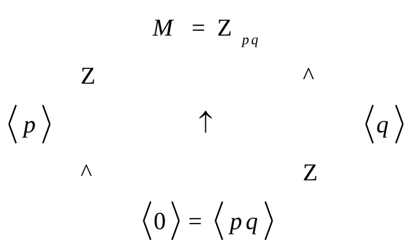
4 Results and Discussion

In this section, we will study the results of the uniserial dimension of the module $M = \mathbb{Z}_m \times \mathbb{Z}_n,\ m, n \in \mathbf{Z}^+$, that obtained from Theorem 3.4 and Corollary 3.8. The discussion will ended by the output of the Python program above.

From Lemma 3.1 and Theorema 3.4, we will get the following example.

Example 4.1

For $M = \mathbb{Z}_n$ where $n = pq$ and p, q are any distinct primes, we will show that $u.s.dim(\mathbb{Z}_n) = 2$ since $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \times \mathbb{Z}_q$. Now let $M = (\mathbb{Z}_n)_{\mathbb{Z}}$. Since any module M contains submodul $\{0\}$ and M, then the lattice of all submodules of M are:



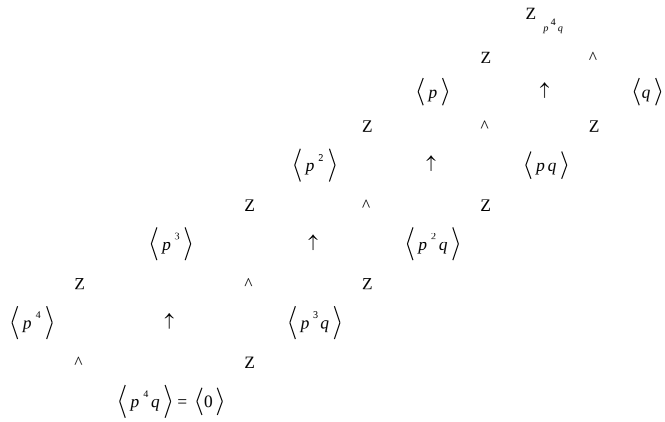
From the lattice above, we get $u.s.dim(\mathbb{Z}_n) = 2$ by Corollary 3.7.

On the other hand, from Lemma 3.1 and Corollary 3.7, we will get the following example.

Example 4.2

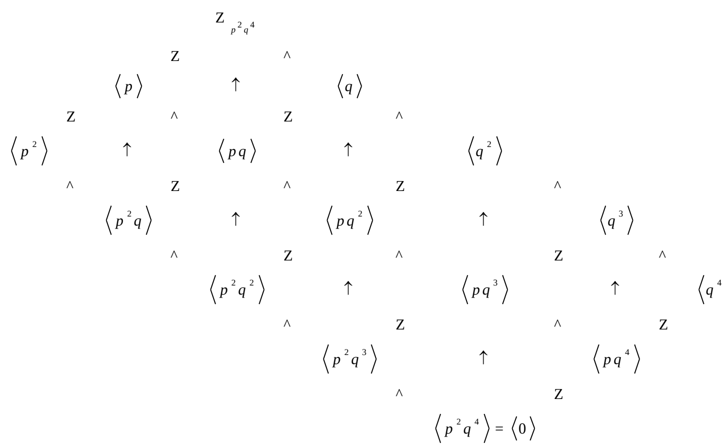
For $M = (\mathbb{Z}_n)_{\mathbb{Z}}$ where $n = p^k q^l,\ k, l \in \mathbb{Z}^+$ and p, q are distinct primes, we will show that $v.dim(\mathbb{Z}_n) = k + l$. Now let $M = (\mathbb{Z}_n)_{\mathbb{Z}}$. Since any module M contains submodul $\{0\}$ and M, then we will get:

1) For $k = 4, l = 1$, we have the lattices of submodules of M as follow:



From the lattice above, we get $u.s.dim\left(\mathbb{Z}_{p^4 q}\right)_{\mathbf{Z}} = 4 + 1 = 5$ by Corollary 3.7.

2) For $k = 2, l = 4$, we have the lattices of submodules of M as follow:



From the lattice above, we get $u.s.dim\left(\mathbb{Z}_{p^2 q^4}\right)_{\mathbf{Z}} = 2 + 4 = 6$ by Corollary 3.7.

Another examples is the following. From Corollary 3.7, and by using Python program above, we can get that $u.s.dim(\mathbb{Z}_{11} \times \mathbb{Z}_{11})_{\mathbf{Z}} = 2$, $u.s.dim(\mathbb{Z}_8 \times \mathbb{Z}_9)_{\mathbf{Z}} = 5$ since $8 = 2^3, 9 = 3^2$, and $u.s.dim(\mathbb{Z}_6 \times \mathbb{Z}_{35}) = 4$ since $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_{35} \cong \mathbb{Z}_5 \times \mathbb{Z}_7$. Output of the result is as follows:

```
================================================
Uniserial Dimension of a Module Z_m x Z_n over Z
------------------------------------------------
Input m:11
Input n:11
We have a module M = Z_ 7 x Z_ 7 over ring Z
Prime factorization of  7 : [11] and 7 : [11]
Therefore, u.s.dim(M) = 2
------------------------------------------------
>>>

================================================
Uniserial Dimension of a Module Z_m x Z_n over Z
------------------------------------------------
Input m:8
Input n:9
We have a module M = Z_ 8 x Z_ 9 over ring Z
Prime factorization of  8 : [2, 2, 2] and 9 : [3, 3]
Therefore, u.s.dim(M) = 5
------------------------------------------------
>>>

================================================
Uniserial Dimension of a Module Z_m x Z_n over Z
------------------------------------------------
Input m:6
Input n:35
We have a module M = Z_ 6 x Z_ 35 over ring Z
Prime factorization of  6 : [2, 3] and 35 : [5, 7]
Therefore, u.s.dim(M) = 4
------------------------------------------------
>>>
```

5 Conclusion

We can determine the uniserial dimension of the module $M = (\mathbb{Z}_m \times \mathbb{Z}_n)_{\mathbb{Z}}, m, n \in \mathbf{Z}^+$ using Python. From the program that has been created, the maximum value of $m, n \in \mathbf{Z}^+$ can be calculated at 1.7×10^{308}. This value corresponds to the upper limit of integers in Python that can be checked by writing the following command.

```
import sys
int(sys.float_info.max)
```

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