On The Convergence Of A Non Linear Method For Ordinary Differential Equations

E. A. Ibijola and R. B. Ogunrinde

Abstract: - *In this paper, we present a comprehensive detail of the convergence theorem for a non-linear method for numerical integration of ordinary differential equations. We prove that the non-linear method is not only stable but convergent.*

***Keywords**: - Initial Value Problem, Stability, Non-Linear, Rational Interpolant*

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1. INTRODUCTION

We shall consider the initial value problem (I. V. P) of the form

$$y' = f(x,y), y(0) = y_{0,} x \in [a,b], y, f \in R \qquad (1)$$

Equation of the form (1) arises in a variety of disciplines. These include Physics (both pure and applied), Engineering, Medicine and even in social sciences. It is a known fact that some of the formulations of government economic policies are based on equation of the form (1). For example, the scientific analysis of population explosion is based on the equation of the form (1). It is a common fact that the spread of modern industrial civilization is as a result of man's ability to solve differential equations. It is also on record that after the industrial evolution, many of the economic theories that emerged are based on the rate of change.

Definition 1.1

A one-step method can be defined as

$$y_{n+1} = y_n + h\Phi(x_n, y_n; h) \qquad (2)$$

Where $\Phi(x_n, y_n; h)$ can be defined as the increment function.

In this paper, we shall assume that the theoretical solution to initial value problem (1) is of the form

$$F(x) = \frac{C}{1 + ae^{\lambda x}} \qquad (3)$$

Where C and a are real constants. We further assume that the theoretical solution $y(x)$ is the approximated by equation (3) in the interval $[x_n, x_{n+1}]$.

- E. A. Ibijola and R. B. Ogunrinde
- Department of Mathematical Sciences, Ekiti State University, Ado Ekiti, Nigeria

With the assumption that $y(x)$ approximate (3), we obtain a one-step method of the form

$$y_{n+1} = \frac{\lambda y_n^2}{\lambda y_n + (e^{\lambda x_n} - 1) h y_n'} \qquad (4)$$

Equation (4) will be the major reference of this work. In short, we shall show that the method represented by (4) is stable and convergent. There had been numerous methods developed to solve initial value problems in ordinary differential equations. Numerical analysis is a branch of Mathematics or Computer Science. It is possible to classify numerical analysis as an art or a science. It is a science in the sense that it is also concerned with the analysis of the developed algorithm with the main objective to identify the most efficient one for a particular problem. (C. F. Fatunla (1982)).

2. CONVERGENT OF A NON-LINEAR METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS.

In this section, we shall formulate a theorem that will guarantee the convergence of method (4). It is necessary to state the following theorems before we proceed further

Theorem 1(Henrici 1962):

Given a differential equation of the form

$y' = f(x,y), y(a) = \eta$. Let $f(x,y)$ be defined and continuous for all points (x,y) in the region defined by $a \le x \le y, -\infty < y < \infty$, a and b, and let there exist a constant L such that for every x, y, y^* with (x,y) and (x, y^*) both in the region defined above

$$\left| f(x,y) - f(x,y^*) \right| \le L \left| y - y^* \right| \qquad (5)$$

If η is any given number, there exist a unique solution $y(x)$ of the initial value problem (1).

The inequality (5) is called a Lipschitz condition and the constant L is called a Lipschitz constant. It is possible to regard the above stated condition as being intermediate between differentiability and continuity, in the sense that if $f(x,y)$ is continuously differentiable with respect to y for all

$f(x,y)$ in the domain under consideration, this implies that $f(x,y)$ satisfies a Lipschitz condition with respect to y for all (x,y) in the domain. (C.F. Fatunla 1988, Lambert 1973a).

If we apply the mean value theorem, we have

$$f(x,y)-f(x,y^*)=\frac{\partial f(x,\bar{y})(y-y^*)}{\partial y} \quad (6)$$

We can achieve (5) by choosing

$$L=\sup_{(x,y)\in Dom}\frac{\partial f(x,\bar{y})}{\partial y} \quad (7)$$

Theorem 2: Henrici (1962)

Let $\Phi(x,y;h)$ be continuous (jointly as a function of its three argument) in the region defined by $x\in[a,b], y\in(-\infty,\infty), 0\le h\le h_0$ where $h_0>0$ and let there exist a constant L such that

$$\left|\Phi(x,y^*;h)-\Phi(x,y;h)\right|\le\left|y^*-y\right| \quad (8)$$

For all $(x,y;h)$ and $(x,y^*;h)$ in the region just defined. Then the relationship

$$\Phi(x,y;0)=f(x,y) \quad (9)$$

Is the necessary and sufficient condition for the convergence of the method defined by the increment Φ.

With the increment function deducted from the formula or scheme.

Proof: Let $\Phi(x,y;0)=g(x,y)$. It is the evident that the function $g(x,y)$ satisfies the condition of theorem 1. For any η, the initial value problem

$$z'=g(x,z), z\ \ a\ \ =\eta \quad (10)$$

has a unique solution $z(x)$. It follows that the point $(x,z(x))\in R$ for $x\in[a,b]$, where R denotes a compact region.

The next step is to show that the numbers z_n defined by $z_0=\eta$ and

$$z_{n+1}=z_n+h\Phi(x_n,z_n,h), n=0,1,2,..., x_n\in[a,b] \quad (11)$$

Converges to $z(x)$. if we subtract the theoretical solution

$$z(x_{n+1})=z(x_n)+h\Delta(x_n,z(x_n);h) \quad (12)$$

From (11) we obtain the error given by

$$e_n=z_n-z(x_n) \quad (13)$$

and

$$e_{n+1}=e_n+h[\Phi(x_n,z_n;h)-\Delta(x_n^*,z(x_n);h)]$$

We now apply the popular mean value theorem to obtain

$$\Delta(x_n,z(x_n);h)=\frac{z(x_{n+1})-z(x_n)}{h}$$

$$=g(x_n+0h,z(x_n+\theta h)) \quad (14)$$

Where $0<\theta<1$.

The function $\Phi(x,y;h)$ is uniformly continuous on the compact set $x\in[a,b], y=z(x), 0\le h\le h_0$. We may then conclude that the quantity represented

$$L_1(h)=\max_{x\in[a,b]}\left|\Phi(x,z(x);h)-\Phi(x,z(x);0)\right| \quad (15)$$

Equation (15) approaches zero as $h\to 0$
Similarly,

$$L_1(h)=\max_{\substack{x\in[a,b]\\0\le k\le h}}\left|g(x,z(x))-g(x,z(x+k)\right| \quad (16)$$

tends to zero as $h\to 0$

We proceed to obtain the estimate

$$\left|e_{n+1}\right|\le\left|e_n\right|+h\left[L\left|e_n\right|+L_1(h)+L_2(h)\right] \quad (17)$$

Inequality [17] is of the form

$$\xi_{n+1}\le A\left|\xi_n\right|+B \quad (18)$$

Where $A=1+hL, B=h(L_1(h)+L_2(h))$, it follows that

$$\left|e_n\right|\le\ \ L_1(h)+L_2(h)\ \ E_L(x_n-a) \quad (19)$$

The expression on the right hand of (19) approaches zero as $h\to 0$, we conclude that

$$\lim_{\substack{h\to 0\\x_n\to x}} z_n=z(x), x\in[a,b] \quad (20)$$

We have succeeded in proving the sufficiency of the consistency condition for convergency. To show the necessity, we assume that the method defined by $\Phi(x,y,h)$ is convergent, but $g(x,y) \neq f(x,y)$ at some point (x,y). Using theorem 1, there exist a η such that the solution $y(t)$ of the initial value problem passes through the point (x,y). The values of z_n previously defined tends $y(t)$ and to the solution $z(t)$. If $z(x) \neq y(x)$ then this leads to a contradiction. If $z(x) = y(x)$ then

$$z' = g(x,y) \neq f(x,y) = y'(x) \qquad (21)$$

This implies that $z(t) \neq y(t)$ leads to contradiction. This completes the proof of the theorem.

Proposition 1: Let $f(x,y)$ be defined as in equation (1), the method

$$y_{n+1} = \frac{\lambda y_n^{(2)}}{\lambda y_n + (e^{\lambda x_n} - 1)hy^{(1)}}, \lambda > 0$$

is stable and convergent.

Proof: Let $y = y(x_n)$ and $l_n = l(x_n)$ denote two different solutions to equation (1) with the initial conditions specified as $y(x_0) = l$ and $l(x_0) = l^*$ such that $\left| l - l^* \right| < \varepsilon, \varepsilon > 0.$

$$l_{n+1} = \frac{\lambda f^{(2)}(x_n^*, l_n)}{\lambda l_n + (e^{\lambda x_n} - 1) f(x_n^*, l_n)} \qquad (22)$$

Equation (21) minus (22) will give us

$$\left| y_{n+1} - l_{n+1} \right| = \lambda \left(\frac{\lambda f^{(2)}(x_n, y_n)}{\lambda y_n + Mf(x_n, y_n)} - \frac{f^{(2)}(x_n^*, l_n)}{\lambda l_n + Mf(x_n^*, l_n)} \right)$$

$$\leq \frac{\lambda}{M} \left| f^{(2)}(x_n, y_n) - f^{(2)}(x_n, l_n) \right|$$

$\leq K \left| l - l^* \right|$, but $\left| l - l^* \right| < \varepsilon, \varepsilon > 0$, we conclude that method (21) is convergent and stable in view of theorem (2)

3. CONCLUSION

The main contribution of this paper is the formulation of proposition 1. We have presented a comprehensive detail of the convergence theorem for a non-linear method

Dedication

This paper is dedicated to the Blessed Memory of Prof. S. O. Fatunla, formerly of University of Benin

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