Some Applicable Methods Of Approximating Basic Trigonometric Functions And Their Inverse Value

Manaye Getu Tsige

Abstract: This paper presents some applicable methods of approximating basic trigonometric functions and their inverse value. Methods are the best choice when a need arise to know first few digits after a decimal point and corresponding angle without spending time for immediate purpose. The ways of approximation are helpful for science and engineering field of study; they can be applied to get immediate solutions for practical problems which might be estimating, comparing and judging while operations of numbers. The assumption stated to carry out this work is; There exists certain function which can satisfies the condition defined as; if the sequence of some domain values within a domain of function forms an arithmetic progression, then the sequence of corresponding range values within a range of function will also forms an arithmetic progression. This assumption leads to the assumed generalized approximate equation and finally to the major findings. The major areas of study to carry out this particular work are arithmetic progression, sine function, cosine function and idea related to trigonometric functions such as trigonometric identities, co terminal angles, reference angle and co function definition. The objective is to contribute additional alternative knowledge to the Mathematical science. The findings of this paper are useful to derive general approximation formulae and other related findings that will be presented in the future.

Index Term: sine function, cosine function, co function, trigonometric identities, arithmetic sequence.

1 INTRODUCTION

Trigonometric functions arise in geometry but they are also applicable to study sound, the motion of pendulum and many other phenomena involving rotation and oscillation [4]. Different approximation techniques have been used to approximate trigonometric functions since along ago. For instance, evaluating trigonometric functions of common angles and quadrant angles by using the properties of triangle and unit circle respectively [10],[7]. Trigonometric table is often used to approximate value and inverse value of trigonometric functions. Sometimes it is possible to use trigonometric identities, sum of two angle formulae, double angle formulae, half angle formulae, product to sum formulae and sum to product formulae provided that angle is reduced to common angles[10] unless otherwise they have no advantage upon evaluating trigonometric functions. There are equations known as Maclaurin series They are given by [9],[4]

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Above equations are also listed in different books [3],[11],[1],[13],[5]. Small angle approximation is given by; for very small x, $\sin x \approx x$ [4]. This paper provides the best alternative methods that should be chose when a need arise to know first few digits after a decimal point and corresponding angle without spending time for immediate purpose.

- *Manaye is currently pursuing masters degree program in Hydraulic Engineering, Arba Minch University, Ethiopia*
  Email: getumanaye100@gmail.com

Methods are helpful for science and engineering field of study; they can be applied to get immediate solutions for practical problems which might be estimating, comparing and judging while operations of numbers. The objective of this paper is to contribute additional alternative knowledge to the Mathematical science. The major areas of study to carry out this particular work are arithmetic progression, sine function, cosine function and idea related to trigonometric functions such as co terminal angles, reference angle and co function definition. Author Hornsby et al., [6] discusses ideas related to trigonometric functions. Co function definition for sine and cosine function is given by

$$\sin\theta = \cos(90 - \beta)$$

In other word, if $\theta + \beta = 90°$, then $\sin\theta = \cos\beta$

Authors (Larson and Hosteller [10], Bronshtein, et al., [2], Solomon [12] discuss about arithmetic sequence. If the difference between successive terms of a sequence is constant, then the sequence is an arithmetic progression. Let $a_1, a_2, a_3, \dots a_4$ is an arithmetic progression, then

$$a_2 - a_1 = a_3 - a_2 = \cdots a_n - a_{n-1} = d$$

If the arithmetic progression has first term a_1 and common difference d, then the n[th] term of the sequence given by

$$a_n = a_1 + d(n-1)$$

2 MATERIALS AND METHODS

Use the following assumption to arrive at assumed generalized approximate equation and use this equation as starting idea to derive provable methods and finally to other related findings.

Assumption: There exists certain function which can satisfy the condition defined within a certain interval of its domain or throughout its domain. It is defined as; if the sequence of some domain values within a domain of function forms an arithmetic progression, then the sequence of corresponding range values within a range of function will also approximately forms an arithmetic progression.

Let x_1 , x_2 , $x_3 \dots x_n$ are some values in its domain (D)

$f(x_1),\ f(x_2),\ f(x_3) \dots f(x_n)$ are corresponding range values in its range(R)

$\{x_1,\ x_2,\ x_3 \dots x_n\} \in D$

$\{f(x_1),\ f(x_2),\ f(x_3) \dots f(x_n)\} \in R$

If $x_1, x_2, x_3 \dots x_n$ is arithmetic sequence, then

$x_2 - x_1 = x_3 - x_2 = \cdots x_n - x_{n-1} = d$

The nth term of the arithmetic sequence given by

$$x_n = x_1 + d(n-1) \qquad (1)$$

$$\Rightarrow n - 1 = \frac{x_n - x_1}{d} \text{ Provided that } d \neq 0 \qquad (2)$$

If $f(x_1),\ f(x_2),\ f(x_3) \dots f(x_n)$ is approximately arithmetic progression, then

$f(x_2) - f(x_1) \approx f(x_3) - f(x_2) \approx \cdots f(x_n) - f(x_{n-1}) \approx m$

The n^{th} term of the sequence is given by

$f(x_n) \approx f(x_1) + m(n-1)$

Assume sine function approximately satisfies above condition within the given interval $[0, x_n]$

$f(x) \approx \sin x$

$$\sin x_n \approx \sin x_1 + m(n-1) \qquad (3)$$

Substitute Eq (2) into (3)

$$\sin x_n \approx \sin x_1 + m\frac{x_n - x_1}{d} \text{ Provided that } d \neq 0$$

$$\Rightarrow m \approx d\frac{\sin x_n - \sin x_1}{x_n - x_1} \text{ Provided that } x_n \neq x_1 \qquad (4)$$

Substitute Eq (1) into (3)

$$\sin(x_1 + d(n-1)) \approx \sin x_1 + m(n-1) \qquad (5)$$

Substitute Eq (4) into (5)

$$\sin(x_1 + d(n-1)) \approx \sin x_1 + d(n-1)\frac{\sin x_n - \sin x_1}{x_n - x_1} \text{ if } x_n \neq x_1 \qquad (6)$$

For non-negative increasing terms of an arithmetic progression, the following inequality holds true.

$0 \le d(n-1) \le x_n$

Having this interval in mind, Let $x = d(n-1)$ and substitute it into Eq (6)

$$\sin(x_1 + x) \approx \sin x_1 + \frac{\sin x_n - \sin x_1}{x_n - x_1}x \text{ if } 0 \le x \le x_n \text{ and } x_n \neq x_1$$

Let $x_1 = 0$

$$\sin(x + 0) \approx \sin 0 + \frac{\sin x_n - \sin 0}{x_n - 0}x \text{ if } 0 \le x \le x_n \text{ and } x_n \neq 0$$

$$\sin x \approx \frac{\sin x_n}{x_n}x \text{ if } 0 \le x \le x_n \text{ and } x_n \neq 0$$

Assume the value of x_n fall within the interval;

$0° < x_n \le 30°$

Then, the assumed generalized approximate equation is given by

$$\sin x \approx \frac{\sin x_n}{x_n}x \text{ if } 0° \le x \le x_n \text{ and } 0° < x_n \le 30° \qquad (7)$$

Use this equation as starting idea to derive provable methods and those methods collectively named as some applicable methods.

2.1 Some applicable methods

Steps to derive provable methods by using assumed generalized approximate equation;

$\sin x \approx \frac{\sin x_n}{x_n}x$ if $0° \le x \le x_n$ and $0° < x_n \le 30°$

Let $x_n = 30°$

$$\sin x \approx \frac{\sin 30}{30}x \text{ if } 0° \le x \le 30°$$

$$\sin x \approx \frac{0.5}{30}x \text{ if } 0° \le x \le 30°$$

$$\sin x \approx \frac{1}{60}x \text{ if } 0° \le x \le 30° \qquad (8)$$

For each value of x within the interval; $0° \le x \le 30°$, the following inequality holds true.

$$0 \le \sin x \le 0.50 \qquad (9)$$

Decimal representation of real numbers: Authors Wrede and Spiegel [14] discuss about decimal representation of real numbers. In order to equally express real number in terms of decimal number, use a few digits and dotted line after a decimal point; $\sqrt{2} = 1.4142 \dots$ Consider decimal representation of real numbers and introduce variables to represent digits after a decimal point.

Definition: To equally express $\sin x$ in terms of decimal number, known digits after a decimal point are denoted by variables A, B, C, D, E, F while other unknown digits are denoted by using dotted line. Having this definition in mind, consider two digits after a decimal point and let digits A and B are known digits after a decimal point, then $\sin x$ approximated by

$$\sin x \approx 0.AB \qquad (10)$$

Substitute Eq (10) into (9)

$$0 \leq 0.AB \leq 0.50 \qquad (11)$$

Correlate Eqs (9), (10) and (11)

$$\sin x \approx 0.AB \ \text{ if } \ 0 \leq 0.AB \leq 0.50 \qquad (12)$$

Substitute Eq (10) into (8)

$$0.AB \approx \frac{x}{60}$$

$$x \approx 60 * 0.AB \qquad (13)$$

Substitute Eq (13) into (12)

$$\sin(0.AB * 60) \approx 0.AB \ \text{ if } 0 \leq 0.AB \leq 0.5 \qquad (14)$$

If the other unknown digits are taken into account and denoted by dotted line, then the approximate Eq (14) can be written as the Eq (15)

$$\sin(0.AB * 60) = 0.AB \ldots.\text{ if } \ 0 \leq 0.AB \leq 0.50 \qquad (15)$$

Mind above definition; variables A and B stand for digits after a decimal point where as dotted line stands for other unknown digits. There are 51- two digits decimal numbers within the interval; $0 \leq 0.AB \leq 0.50$ and hence it is now very easy to verify equation (15) whether it holds true or not for all possible two digits decimal numbers. Use trigonometric table to crosscheck equation (15) for each two digits decimal number.

$$\sin(0.AB * 60) = 0.AB \ldots.\text{ if } \ 0 \leq 0.AB \leq 0.50$$

Let $0.AB = 0.38$ provided that $0 \leq 0.AB \leq 0.50$

$$0 \leq 0.38 \leq 0.50$$

$$\sin(0.38 * 60) = 0.38 \ldots$$

$$\sin 22.8° = 0.38 \ldots$$

Consider co function definition [7]

If $x + y = 90°$, then $\sin x = \cos y$

$$\cos(90 - 0.AB * 60) = 0.AB \ldots$$

$$\cos(90 - 0.38 * 60) = 0.38 \ldots$$

$$\cos 67.2° = 0.38 \ldots$$

Summarize it by constructing table for all two digits decimal numbers (0. AB)

Table 1. Digits versus corresponding sine value

	A	$x = \sin(0.AB * 60) = 0.AB \ldots$									
B	x	0	1	2	3	4	5	6	7	8	9
	0	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
	1	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19
	2	0.20	0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28	0.29
	3	0.30	0.31	0.32	0.33	0.34	0.35	0.36	0.37	0.38	0.39
	4	0.40	0.41	0.42	0.43	0.44	0.45	0.46	0.47	0.48	0.49
	5	0.50									

Table 1 shows Eq (15) holds true for all two digits decimal numbers. Equations (8) and (16) satisfy co function definition.

$$\cos(90 - x) \approx \tfrac{1}{60}x \ \text{ if } \ 0° \leq x \leq 30° \qquad (16)$$

If the negative and other co terminal angles are taken into account, then the Eq (15) can be modified as the method one and state the equations (8) and (16) as the first approximation method. The maximum error of approximation can be calculated by using known digits after decimal point.

2.1.1 Method one

If the $\mathrm{A}, \mathrm{B}, \ldots$ stand for digits after a decimal point and the angle is expressed in degree unit, then the following equations expressed in decimal number hold true.

$$\sin(\pm 0.AB * 60 + 360n) = \pm 0.AB \ldots \text{ if } \ 0 \leq 0.AB \leq 0.50$$

$$\cos(\pm(90 - 0.AB * 60) + 360n) = 0.AB \ldots \text{ if } \ 0 \leq 0.AB \leq 0.50$$

Where n is integer

The maximum error of approximation is less than 0.009

2.1.2 First approximation method

$$\sin(x + 360n) \approx \frac{1}{60}x \ \text{ if } -30° \leq x \leq 30°$$

$$\cos(90 - x + 360n) \approx \frac{1}{60}x \ \text{ if } -30° \leq x \leq 30°$$

The maximum error of approximation is less than 0.009 All other methods can be derived in the same way as above methods derived and each method can be verified in the same manner. And therefore, it is unnecessary to repeat steps hereafter. Take $x_n = 15°$ to derive method two and second approximation method.

2.1.3 Method two
If the A, B, C stand for digits after a decimal point and the angle is expressed in degree unit, then the following equations expressed in decimal number hold true.

$$\sin(\pm 0.ABC * \frac{60}{\sqrt{6}-\sqrt{2}} + 360n) \approx \begin{cases} \pm 0.ABC \text{ if } 0 \le 0.ABC \le 0.258 \text{ and } A \ne 1 \\ \pm(0.ABC + 0.001) \text{ if } 0 \le 0.ABC \le 0.258 \text{ and } A = 1 \end{cases}$$

$$\mathrm{Cos}(\pm(90 - 0.ABC * \frac{60}{\sqrt{6}-\sqrt{2}}) + 360n) \approx \begin{cases} 0.ABC \text{ if } 0 \le 0.ABC \le 0.258 \text{ and } A \ne 1 \\ 0.ABC + 0.001 \text{ if } 0 \le 0.ABC \le 0.258 \text{ and } A = 1 \end{cases}$$

Equations hold true for all 259 three digits decimal numbers $(0.ABC)$ within the interval; $0 \le 0.ABC \le 0.258$ The maximum error of approximation is less than 0.0009

2.1.4 Second approximation method

$$\sin(x + 360n) \approx \frac{\sin 15}{15}x \approx \frac{\sqrt{6}-\sqrt{2}}{60}x \text{ if } -15° \le x \le 15°$$

$$\cos(90 - x + 360n) \approx \frac{\sin 15}{15}x \approx \frac{\sqrt{6}-\sqrt{2}}{60}x \text{ if } -15° \le x \le 15°$$

The maximum error of approximation is less than 0.0009
Next, take $x_n = 5°$ to derive method three and third approximation method

2.1.5 Method three
If the $A, B, C, D \ldots$ stand for digits after a decimal point and the angle is expressed in degree unit, then the following equations expressed in decimal number hold true.

$$\sin(\pm 0.ABCD * \frac{5}{\sin 5} + 360n) = \pm 0.ABCD \ldots \text{ if } 0 \le 0.ABCD \le 0.0871$$

$$\cos\left(\pm(90 - 0.ABCD * \frac{5}{\sin 5}) + 360n\right) = 0.ABCD \ldots \text{ if } 0 \le 0.ABCD \le 0.0871$$

Equations hold true for all 872 four digits decimal numbers within the interval; $0 \le 0.ABCD \le 0.0871$ The maximum error of approximation is less than 0.00009

2.1.6 Third approximation method

$$\sin(x + 360n) \approx \frac{\sin 5}{5}x \text{ if } -5° \le x \le 5°$$

$$\cos(90 - x + 360n) \approx \frac{\sin 5}{5}x \text{ if } -5° \le x \le 5°$$

The maximum error of approximation is less than 0.00009
Next, take $x_n = 1°$ to derive method four and fourth approximation method

2.1.7 Method four
If the $A, B, C, D, E, F \ldots$ stand for digits after a decimal point and the angle is expressed in degree unit, then the following equations expressed in decimal number hold true.

$$\sin(\pm 0.ABCDEF * \frac{1}{\sin 1} + 360n) = \pm 0.ABCDE \ldots \text{ if } 0 \le 0.ABCDEF \le 0.017452$$

$$\cos(\pm(90 - 0.ABCDEF * \frac{1}{\sin 1}) + 360n) = 0.ABCDEF \ldots \text{ if } 0 \le 0.ABCDEF \le 0.017452$$

Equations hold true for all 17453 six digits decimal numbers within the interval; $0 \le 0.ABCDEF \le 0.017452$ The maximum error of approximation is less than 0.0000009

2.1.8 Fourth approximation method

$$\sin(x + 360n) \approx x \sin 1 \text{ if } -1 \le x \le 1°$$

$$\cos(90 - x + 360n) \approx x \sin 1 \text{ if } -1 \le x \le 1°$$

The maximum error of approximation is less than 0.0000009

3 RESULTS AND DISCUSSION

The main results of this paper are method one, two, three, four and corresponding approximation methods. They are collectively named as some applicable methods and they are used to compute value and inverse of basic trigonometric functions. For instance, in the case of inverse value computation, method one, two and three can also be written by

$$\left.\begin{aligned} &\sin^{-1}(\pm 0.AB) \approx \pm 0.AB * 60 + 360n \text{ if } 0 \le 0.AB \le 0.50 \\ &\cos^{-1}(0.AB) \approx \pm(90 - 0.AB * 60) + 360n \text{ if } 0 \le 0.AB \le 0.50 \end{aligned}\right\}$$

$$\left.\begin{aligned} &\sin^{-1}(\pm 0.ABC) \approx \pm 0.ABC \frac{60}{\sqrt{6}-\sqrt{2}} + 360n \text{ if } 0 \le 0.ABC \le 0.258 \\ &\cos^{-1}(0.ABC) \approx \pm\left(90 - 0.ABC * \frac{60}{\sqrt{6}-\sqrt{2}}\right) + 360n \text{ if } 0 \le 0.ABC \le 0.258 \end{aligned}\right\}$$

$$\left.\begin{aligned} &\sin^{-1}(\pm 0.ABCD) \approx \pm 0.ABCD * 57.368566 + 360n \text{ if } 0 \le 0.ABCD \le 0.0871 \\ &\cos^{-1}(0.ABCD) \approx \pm(90 - 0.ABCD * 57.368566) + 360n \text{ if } 0 \le 0.ABCD \le 0.0871 \end{aligned}\right\}$$

Within common interval, the error of approximation gets smaller and smaller from the first up to the fourth approximation method. This implies that the assumed generalized approximate equation is very interesting and supposed to be holds true. And therefore, Sine function approximately satisfies above condition within the interval; $[-30°, 30°]$ provided that the negative angles are taken into consideration.

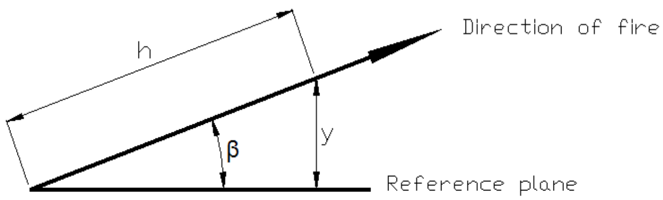
3.1 APPLICATIONS

The results of this research work have applications in Engineering, Military institution, Education.

3.1.1 Military institution/Engineering areas
To set angle based on target of interest. Compares to all other trigonometric functions evaluation techniques found in literature, this paper contains expression in the form of

simple fraction that it enables to set angle by using a value of numerator and denominator.

Example 1: Target direction of fire at $\beta \approx 20°$ (near to $20°$ but not exactly $20°$)



Consider the first approximation method

$$\sin\beta \approx \frac{\beta}{60} \text{ if } 0 \leq \beta \leq 30°$$

$\sin 20° \approx \frac{20}{60}$ if $0 \leq 20 \leq 30°$ (it is ok)

$$\sin 20° \approx \frac{20}{60} \approx \frac{1}{3}$$

Use right angle triangle definition of Trigonometric functions.

$$\sin\beta \approx \frac{y}{h} \approx \frac{1}{3}$$

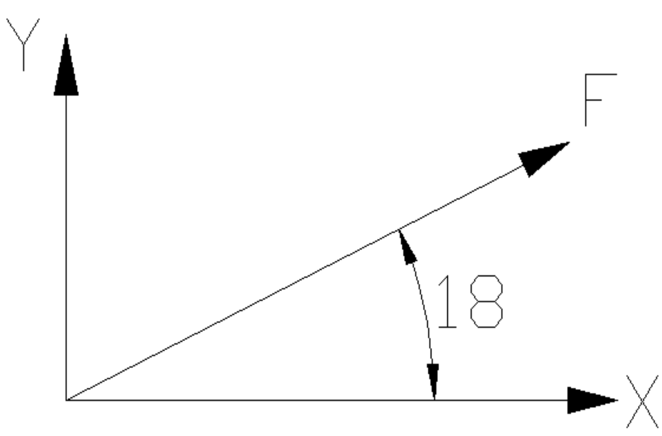
Take $y = 1$ and $h = 3$

Set angle by measuring one unit upward and three units along the direction of fire.

3.1.2 Education

Methods minimize effort while teaching different science.

Example 2: calculate y component of force F= 50 N



$F_y = F \sin 18°$

Approximate $\sin 18°$ by using the first approximation method.

$$\sin x \approx \frac{x}{60} \text{ if } -30° \leq x \leq 30°$$

$$\sin 18 \approx \frac{18}{60} \approx 0.3$$

$$F_y \approx 50 * 0.3 \approx 15N$$

4 CONCLUSIONS

In general, the limitation of this paper is maybe approximation becomes possible only within the given interval. It provides foot step to derive general approximation formulae and other related findings that will be presented in the future.

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