Fourier Cosine Transform Method For Solving The Elasticity Problem Of Point Load On An Elastic Half Plane

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Abstract— The Fourier cosine transform method was used in this work to obtain general solutions for stresses and displacement fields in homogeneous, isotropic linear elastic soil of semi-infinite extent due to a point load applied vertically at the origin of the half plane. A stress-based formulation was used. Boussinesq-Papkovich potential functions were used to express the displacement components, strains, and stresses. Fourier cosine transformation of the biharmonic stress compatibility equation was done to obtain suitable bounded stress functions for the problem. Stresses were similarly expressed in the Fourier cosine transform space variable, by Fourier cosine transformation. Enforcement of stress boundary conditions were used to obtain the unknown constants of the stress functions. Inversion of the Fourier cosine transforms of the stresses yielded the general expressions for the stress fields in the physical domain two dimensional Cartesian variables. The strain fields were obtained using the kinematic relations of infinitesimal elasticity. The displacement fields were determined by integration of the strain-displacement equations. The expressions for displacement and stress fields obtained were identical with solutions in literature obtained using Airy's stress potential functions.

Index Terms— Biharmonic stress compatibility equation, Boussinesq-Papkovich potential functions, displacement fields, elastic half plane problem, Fourier cosine transform method, Stress fields.

1 INTRODUCTION

THE determination of stress and displacement fields in an elastic half plane due to point or distributed loads acting on the surface or inside the half plane is a theory of elasticity problem; and can be treated as a two dimensional particularization of the problem of elastic half space [1 – 6]. Such problems are frequently encountered in the analysis and design of geotechnical and foundation structures. The assumptions made as to the behaviour of the elastic half plane material define the complexity or otherwise of the mathematical problems involved. The half plane material may be assumed linear elastic or non-linear elastic, isotropic or non-isotropic (anisotropic), homogeneous or non-homogeneous. Anisotropic, heterogenous non-linear assumptions of the elastic half plane problem involve mathematically rigorous and demanding formulations and solutions [7 – 10]. In this work, the elastic half plane is assumed to be homogeneous, isotropic and linear-elastic. Elasticity problems involving half plane and half-space regions are governed by the fundamental equations of the theory of elasticity; namely: the three differential equations of equilibrium, the six generalised stress-strain equations, and the six strain-displacement equations subject to the compatibility equations and the traction and deformation boundary conditions [11 – 15]. Due to the high number of equations and their complexities, the closed form solutions of elastic half plane/space problems are difficult to obtain [11 – 15]. Three general methods found in the technical literature for the simplification of the elastic half-plane (space) problems are the displacement method, the stress method and the mixed method [3 – 6, 13]. In displacement formulation, the governing field equations are expressed in terms of displacement field components as the primary unknown variables by combining the stress equilibrium equations, the stress-strain relations, and the strain-displacement equations. The equations of the displacement formulation for three dimensional elasticity problems are a system of three partial differential equations in terms of three unknown displacement field components, and thus can be solved. In stress formulation, the governing field equations are expressed in terms of the unknown stress components as the primary unknowns. In solving problems of stress-based formulations, the stresses are first determined and then the displacement field components determined using the stress-strain laws and the strain-displacement relations. The displacement components are three, while the strain components are six, resulting in a system of six equations in three unknown displacement components. Three dimensional elasticity stress-based problems can only be solved by seeking additional equations. In stress-based formulation, compatibility of strains are used to generate additional equations. Two sets of strain compatibility equations can be derived; the first set expresses the shear strains in terms of normal strain while the second set expresses the normal strains in terms of shear strain [3 – 6]. The strains are expressed in terms of stresses using the stress-strain relations to obtain the strain-compatibility equations written in terms of stresses. The resulting equations are called stress compatibility equations or the Beltrami-Michell equations. The theoretical foundations of stress-based formulations of elasticity were laid by Airy, Michell, Love, Morera, Beltrami, Maxwell, Boussinesq, Papkovich and numerous other scholars [16 – 19]. In the mixed formulation, the governing domain equations are expressed such that the unknowns are some components of stress, and some displacement components. The analytical simplifications evident in the stress-based and displacement-based methods have led researchers to derive stress functions, and displacement functions that solve the governing equations of each method. The stress and displacement functions further simplify the solutions of elasticity problems in that they are reduced to finding appropriate stress and displacement functions in each particular problem to be solved [20 – 27]. In this present study, the Bousinesq-Papkovich displacement (stress) potential functions are used in the method of Fourier cosine transformation to obtain the stress fields and displacement fields in a half plane due to a vertically applied point load P_0 acting at the origin in the positive z direction. This problem was first solved by Flammant who used Airy's stress function method.

1.1 Research Aim and Objectives

The research aim is to use the Fourier cosine transform method for solving the elasticity problem of point load applied at the origin of an elastic half plane.

The objectives are:
(i) to present a stress formulation of the problem as a biharmonic differential equation in terms of stress potential functions
(ii) to find the Fourier cosine transform of the biharmonic stress compatibility equation, and thus obtain bounded stress functions for the problem.
(iii) to find the stress and strain fields in the Fourier cosine transform space
(iv) to enforce the boundary conditions and obtain unknown constants of the stress functions
(v) to use the inverse Fourier cosine transformation to obtain the stresses, strains and displacements in the physical domain space variable.

2 Research Problem: the point load on an elastic half plane

The point load on an elastic half plane is the problem of determining the stress fields and displacement fields at any point $A(x, z)$ or $A(r, z)$ in the half plane due to a single point load P_0 acting at the origin (0, 0) on the surface of a linear elastic half plane material of semi-infinite extent, where $-\infty \leq x \leq \infty$, $0 \leq z \leq \infty$, and $r^2 = x^2 + z^2$. The problem is shown in Figure 1.

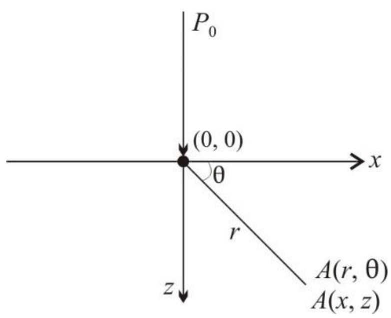

Fig. 1. Point load on an elastic half plane

Mathematically, the problem is to determine the stress fields and displacement fields satisfying the stress compatibility equations, the stress-strain equations, and the strain-displacement equations as well as the boundary conditions. The boundary conditions are:

$$\sigma_{zz}(x, z = 0) = -P_0\delta(x) \quad (1)$$

where $\delta(x)$ is the Dirac delta function, σ_{zz} is the normal stress field.

$$\tau_{xz}(x, z = 0) = 0 \quad (2)$$

where τ_{xz} is the shear stress field

Equation (2) represents the shear stress free condition on the *xy* plane (i.e. *z* = 0)_. The boundedness of stresses require that the stresses tend to zero as $|x| \to \infty$ i.e. $x \to \infty$ or $x \to -\infty$.

3 Methodology

The Boussinesq-Papkovich displacement potential functions $\phi(x, z)$ and $\psi(x, z)$ are scalar fields of the two dimensional Cartesian coordinates *x*, *z* from which the displacement components are derivable as follows [20]:

$$u_x(x, z) = u(x, z) = -\frac{1}{4(1-\mu)}\frac{\partial\phi}{\partial x}(x, z) \quad (3)$$

$$w(x, z) = u_z(x, z) = \psi(x, z) - \frac{1}{4(1-\mu)}\frac{\partial\phi}{\partial z}(x, z) \quad (4)$$

where $\nabla^2\phi(x, z) = 0$ (5)

$$\nabla^2\psi(x, z) = 0 \quad (6)$$

and, $\nabla^4\phi(x, z) = 0$ (7)

u_x, or *u* is the displacement field component in the *x* direction, *w* or u_z is the displacement field component in the *z* direction, μ is the Poisson's ratio of the elastic half plane material, and∇^2 denotes the Laplacian∇^4 is the biharmonic operator. $\phi(x, z)$ is a scalar (field) function of *x* and *z*.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \quad (8)$$

3.1 Strain Fields

The strain fields are expressed in terms of the Boussinesq-Papkovich displacement potential functions using the strain-displacement relations of small-displacement (infinitesimal displacement) assumptions in the theory of elasticity as [20]:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{4(1-\mu)}\frac{\partial\phi}{\partial x}(x, z)\right) = -\frac{1}{4(1-\mu)}\frac{\partial^2\phi}{\partial x^2}(x, z) \quad (9)$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial x} = \frac{\partial}{\partial z}\left(\psi(x, z) - \frac{1}{4(1-\mu)}\frac{\partial\phi}{\partial z}(x, z)\right)$$

$$= \frac{\partial\psi}{\partial z}(x, z) - \frac{1}{4(1-\mu)}\frac{\partial^2\phi}{\partial z^2}(x, z) \quad (10)$$

$$\gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial}{\partial z}\left(-\frac{1}{4(1-\mu)}\frac{\partial\phi(x, z)}{\partial x}\right)$$

$$+\frac{\partial}{\partial x}\left(\psi(x, z) - \frac{1}{4(1-\mu)}\frac{\partial\phi(x, z)}{\partial z}\right) \quad (11)$$

$$\gamma_{xz} = \frac{\partial\psi(x, z)}{\partial x} - \frac{1}{2(1-\mu)}\frac{\partial^2\phi(x, z)}{\partial x\partial z} \quad (12)$$

$$\varepsilon_{xz} = \frac{1}{2}\gamma_{xz} = -\frac{1}{4(1-\mu)}\left(\frac{\partial^2\phi(x, z)}{\partial x\partial z} - 2(1-\mu)\frac{\partial\psi(x, z)}{\partial x}\right) \quad (13)$$

3.2 Stress Fields

The stress fields are found from the generalised Hooke's stress-strain relations expressed in terms of Lamé constants. The stress-strain laws are [20]:

$$\sigma_{xx} = 2G\varepsilon_{xx} + \lambda\varepsilon_v \quad (14)$$

$$\sigma_{yy} = \lambda\varepsilon_v \quad (15)$$

$$\sigma_{zz} = 2G\varepsilon_{zz} + \lambda\varepsilon_v \quad (16)$$

$$\tau_{xz} = G\gamma_{xz} = 2G\varepsilon_{xz} \quad (17)$$

$$\tau_{xy} = 0 \quad (18)$$

$$\tau_{yz} = 0 \quad (19)$$

where ε_v is the volumetric strain, σ_{xx}, σ_{yy}, and σ_{zz} are normal stresses and τ_{xz}, τ_{xy}, τ_{yz} are shear stresses, *G* is the shear modulus of elasticity, λ, the Lamé's constant is related to the elastic constants *G* and μ as follows:

$$\lambda = \frac{2\mu G}{1-2\mu} \quad (20)$$

The volumetric strain ε_v is expressed in terms of the Boussinesq-Papkovich potential function $\psi(x,z)$ as:

$$\varepsilon_v = \frac{1-2\mu}{2(1-\mu)}\frac{\partial\psi}{\partial z}(x,z) \quad (21)$$

The stress fields are expressed in terms of the Boussinesq-Papkovich potential functions as follows [20]:

$$\sigma_{xx}(x,z) = \frac{-G}{2(1-\mu)}\left(\frac{\partial^2\phi}{\partial x^2} - 2\mu\frac{\partial\psi}{\partial z}\right) \quad (22)$$

$$\sigma_{zz}(x,z) = \frac{-G}{2(1-\mu)}\left(\frac{\partial^2\phi}{\partial x^2} - 2(2-\mu)\frac{\partial\psi}{\partial z}\right) \quad (23)$$

$$\tau_{xz}(x,z) = \frac{-G}{2(1-\mu)}\left(\frac{\partial^2\phi(x,z)}{\partial x\partial z} - 2(1-\mu)\frac{\partial\psi(x,z)}{\partial x}\right) \quad (24)$$

It is thus observed that the Boussinesq-Papkovich potential functions defined in Equations (3) and (4) as displacement harmonic functions could also be defined as stress potential functions using Equations (22-24); which present the stress fields (σ_{xx}, σ_{zz} and τ_{xz}) as derivable from the potential functions $\phi(x,z)$ and $\psi(x,z)$.

4 RESULTS

4.1 Stress Compatibility Equation

The stress compatibility equation is expressed as the biharmonic equation in $\phi(x,z)$ given by:

$$\nabla^4\phi(x,z) = \nabla^2\nabla^2\phi(x,z) = 0 \quad (25)$$

where ∇^4 is the biharmonic operator, defined as:

$$\nabla^4 = (\nabla^2)^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)^2 = \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial z^2} + \frac{\partial^4}{\partial z^4}\right) \quad (26)$$

4.2 Fourier Cosine Transform Solution for a Suitable Stress Function

The Fourier cosine transform technique is used to find a suitable stress function for the half plane problem. The biharmonic stress compatibility equation is solved using the Fourier cosine transform method. Applying the Fourier cosine transformation with respect to the *x*-coordinate to the biharmonic stress compatibility equation yields:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial z^2} + \frac{\partial^4\phi}{\partial z^4}\right)\cos\beta x\,dx = 0 \quad (27)$$

where β is the Fourier cosine transform parameter.

$$\int_{-\infty}^{\infty}\left(\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial z^2} + \frac{\partial^4\phi}{\partial z^4}\right)\cos\beta x\,dx = 0 \quad (28)$$

$$\int_{-\infty}^{\infty}\frac{\partial^4\phi}{\partial x^4}\cos\beta x\,dx + 2\frac{\partial^2}{\partial z^2}\int_{-\infty}^{\infty}\frac{\partial^2\phi}{\partial x^2}\cos\beta x\,dx + \frac{\partial^4}{\partial z^4}\int_{-\infty}^{\infty}\phi(x,z)\cos\beta x\,dx = 0 \quad \ldots(29)$$

$$\beta^4\bar{\phi}(\beta,z) - 2\beta^2\frac{d^2\bar{\phi}(\beta,z)}{dz^2} + \frac{d^4}{dz^4}\bar{\phi}(\beta,z) = 0 \quad (30)$$

where $\bar{\phi}(\beta,z) = \int_{-\infty}^{\infty}\phi(x,z)\cos\beta x\,dx$ (31)

$\bar{\phi}(\beta,z)$ is the Fourier cosine transform of $\phi(x,z)$ with respect to the *x* variable.

The fourth order biharmonic stress compatibility equation is thus transformed to the fourth order homogeneous ordinary differential equation in $\bar{\phi}(\beta,z)$ which is the stress function in the Fourier cosine transform space. The general solution to Equation (30) is obtained using differential operator methods, auxiliary polynomial technique or trial functions methods as:

$$\bar{\phi}(\beta,z) = (c_1 + c_2\beta z)\exp(-\beta z) + (c_3 + c_4\beta z)\exp(\beta z) \quad (32)$$

where c_1, c_2, c_3 and c_4 are the four constants of integration.

The boundary condition requiring bounded stress fields as $z \to \infty$ requires that $\bar{\phi}(\beta,z)$ be bounded and finite as $z \to \infty$.

Thus, for bounded $\bar{\phi}(\beta,z)$ as $z \to \infty$,

$c_3 = 0$ (33)

$c_4 = 0$ (34)

Hence bounded stress field in the Fourier cosine transform space is obtained as:

$$\bar{\phi}(\beta,z) = (c_1 + c_2\beta z)\exp(-\beta z) \quad (35)$$

The Boussinesq-Papkovich stress (or displacement) potential functions in the Fourier cosine transform space can then be found as:

$$\bar{\psi}(\beta,z) = c_2\beta\,\exp(-\beta z) = c_2\beta e^{-\beta z} \quad (36)$$

and $\bar{\phi}(\beta,z)$ as given by Equation (35).

By inversion, the Boussinesq-Papkovich stress potential functions are obtained in terms of the physical domain space variables as:

$$\phi(x,z) = \int_0^{\infty}\bar{\phi}(\beta,z)\,\cos\beta x\,d\beta \quad (37)$$

$$\phi(x,z) = \int_0^{\infty}(c_1 + c_2\beta z)e^{-\beta z}\cos\beta x\,d\beta \quad (38)$$

$$\psi(x,z) = \int_0^{\infty}\bar{\psi}(\beta,z)\,\cos\beta x\,d\beta \quad (39)$$

$$\psi(x,z) = \int_0^{\infty}c_2\beta e^{-\beta z}\cos\beta x\,d\beta \quad (40)$$

4.3 Fourier Cosine Transformation of Stress Fields

Application of the Fourier cosine transformation to the stress fields yield the following expressions for the stress fields in the Fourier cosine transform space.

$$\bar{\sigma}_{xx}(\beta,z) = \int_{-\infty}^{\infty}\frac{-G}{2(1-\mu)}\left(\frac{\partial^2\phi}{\partial x^2} - 2\mu\frac{\partial\psi}{\partial z}\right)\cos\beta x\,dx \quad (41)$$

$$\bar{\sigma}_{xx}(\beta,z) = \frac{-G}{2(1-\mu)}\int_{-\infty}^{\infty}\frac{\partial^2\phi}{\partial x^2}\cos\beta x\,dx + \frac{G\mu}{1-\mu}\int_{-\infty}^{\infty}\frac{\partial\psi}{\partial z}\cos\beta x\,dx \quad (42)$$

$$\bar{\sigma}_{xx}(\beta,z) = \frac{-G}{2(1-\mu)}(i\beta)^2\int_{-\infty}^{\infty}\phi(x,z)\cos\beta x\,dx + \frac{G\mu}{1-\mu}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\psi(x,z)\cos\beta x\,dx \quad \ldots(43)$$

$$\bar{\sigma}_{xx}(\beta,z) = \frac{G\beta^2}{2(1-\mu)}\bar{\phi}(\beta,z) + \frac{G\mu}{1-\mu}\frac{\partial}{\partial z}\bar{\psi}(\beta,z) \quad (44)$$

$$\bar{\sigma}_{xx}(\beta,z) = \frac{G\beta^2}{2(1-\mu)}(c_1 + c_2\beta z)e^{-\beta z} + \frac{G\mu}{1-\mu}\frac{\partial}{\partial z}c_2\beta e^{-\beta z} \quad (45)$$

$$\bar{\sigma}_{xx}(\beta,z)=\frac{G\beta^2}{2(1-\mu)}(c_1+c_2\beta z)e^{-\beta z}-\frac{G\mu}{1-\mu}c_2\beta^2e^{-\beta z} \quad (46)$$

Similarly,

$$\bar{\sigma}_{zz}(\beta,z)=\frac{-G}{2(1-\mu)}\left\{\frac{\partial^2}{\partial z^2}\bar{\phi}(\beta,z)-2(2-\mu)\frac{\partial}{\partial z}\bar{\psi}(\beta,z)\right\} \quad \ldots(47)$$

$$\bar{\sigma}_{zz}(\beta,z)=\frac{-G}{2(1-\mu)}\left\{\frac{\partial^2}{\partial z^2}(c_1+c_2\beta z)e^{-\beta z}+2(2-\mu)c_2\beta^2e^{-\beta z}\right\} \quad \ldots(48)$$

$$\bar{\sigma}_{zz}(\beta,z)=\frac{-G}{2(1-\mu)}\{c_1\beta^2e^{-\beta z}-c_2\beta^2(e^{-\beta z}+e^{-\beta z}-\beta z e^{-\beta z}) +2(2-\mu)c_2\beta^2e^{-\beta z}\} \quad (49)$$

$$\bar{\sigma}_{zz}(\beta,z)=\frac{-G}{2(1-\mu)}\left\{\left((c_1\beta^2-c_2\beta^2(2-\beta z)\right)e^{-\beta z}+2(2-\mu)c_2\beta^2e^{-\beta z}\right\} \quad \ldots(50)$$

$$\bar{\tau}_{xz}(\beta,z)=\frac{-G}{2(1-\mu)}\frac{\partial}{\partial z}(i\beta)\bar{\phi}(\beta,z)+G(i\beta)\bar{\psi}(\beta,z) \quad (51)$$

$$\bar{\tau}_{xz}(\beta,z)=\frac{-Gi\beta}{2(1-\mu)}\frac{\partial}{\partial z}(c_1+c_2\beta z)e^{-\beta z}+Gi\beta c_2\beta e^{-\beta z} \quad (52)$$

$$\bar{\tau}_{xz}(\beta,z)=\frac{-Gi\beta}{2(1-\mu)}\left\{\left(-\beta c_1e^{-\beta z}+c_2\beta(e^{-\beta z}+z(-\beta e^{-\beta z}))\right)\right\}+Gi\beta c_2\beta e^{-\beta z} \quad \ldots(53)$$

$$\bar{\tau}_{xz}(\beta,z)=\frac{-Gi\beta}{2(1-\mu)}\left\{-\beta c_1e^{-\beta z}+c_2\beta(1-z\beta)e^{-\beta z}\right\}+Gi\beta^2c_2e^{-\beta z} \quad \ldots(54)$$

$$\bar{\tau}_{xz}(\beta,z=0)=\frac{-Gi\beta}{2(1-\mu)}(-\beta c_1+c_2\beta)+Gi\beta^2c_2=0 \quad (55)$$

$$\frac{-Gi\beta^2}{2(1-\mu)}(-c_1+c_2)+Gi\beta^2c_2=0 \quad (56)$$

Solving, Equation (56) we obtain:

$$c_1=-(1-2\mu)c_2 \quad (57)$$

$$\bar{\sigma}_{zz}(\beta,z=0)=\frac{-G}{2(1-\mu)}\left((c_1\beta^2-2c_2\beta^2)+2(2-\mu)c_2\beta^2\right) \quad (58)$$

$$\bar{\sigma}_{zz}(\beta,z=0)=\frac{-G}{2(1-\mu)}\left(-(1-2\mu)c_2\beta^2-2c_2\beta^2+2(2-\mu)c_2\beta^2\right) \quad \ldots(59)$$

$$\bar{\sigma}_{zz}(\beta,z=0)=\frac{-G}{2(1-\mu)}(-1+2\mu-2+4-2\mu)c_2\beta^2 \quad (60)$$

$$\bar{\sigma}_{zz}(\beta,z=0)=\frac{-G}{2(1-\mu)}c_2\beta^2=\frac{-P_0}{\pi} \quad (61)$$

$$c_2\beta^2=\frac{2(1-\mu)P_0}{G\pi} \quad (62)$$

$$c_2=\frac{2(1-\mu)P_0}{G\pi\beta^2} \quad (63)$$

$$c_1=-(1-2\mu)c_2=\frac{-2(1-\mu)(1-2\mu)P_0}{\pi\beta^2G} \quad (64)$$

The stress functions are thus completely determined. The stresses are obtained as:

$$\bar{\sigma}_{xx}(\beta,z)=\frac{G\beta^2}{2(1-\mu)}\left(\frac{-2(1-\mu)(1-2\mu)P_0}{\pi\beta^2G}+\frac{2(1-\mu)P_0}{\pi\beta^2G}\beta z\right)e^{-\beta z} -\frac{G\mu\beta^2}{1-\mu}\frac{2(1-\mu)P_0}{\pi\beta^2G}e^{-\beta z} \quad (65)$$

$$\bar{\sigma}_{xx}(\beta,z)=\left(\frac{-(1-2\mu)P_0}{\pi}+\frac{P_0\beta z}{\pi}-\frac{2\mu P_0}{\pi}\right)e^{-\beta z} \quad (66)$$

$$\bar{\sigma}_{xx}(\beta,z)=\frac{P_0}{\pi}(\beta z+2\mu-1+2\mu)e^{-\beta z}=\frac{P_0}{\pi}(\beta z-1)e^{-\beta z} \quad (67)$$

By inversion, the stress field $\sigma_{xx}(x,z)$ is obtained in the physical domain variables as:

$$\sigma_{xx}(x,z)=\int_0^\infty\frac{P_0}{\pi}(\beta z-1)e^{-\beta z}\cos\beta x\,d\beta \quad (68)$$

$$\sigma_{xx}(x,z)=-\frac{P_0}{\pi}\int_0^\infty(1-\beta z)e^{-\beta z}\cos\beta x\,d\beta \quad (69)$$

$$\sigma_{xx}(x,z)=-\frac{2P_0x^2z}{\pi r^4}=-\frac{2P_0x^2z}{\pi(x^2+z^2)^2} \quad (70)$$

Similarly,

$$\bar{\sigma}_{zz}(\beta,z)=\frac{-G\beta^2}{2(1-\mu)}(c_1-c_2(2-\beta z)+2(2-\mu)c_2)e^{-\beta z} \quad (71)$$

$$\bar{\sigma}_{zz}(\beta,z)=\frac{-G\beta^2}{2(1-\mu)}(c_1+c_2(2-2\mu+\beta z))e^{-\beta z} \quad (72)$$

$$\bar{\sigma}_{zz}(\beta,z)=\frac{-G\beta^2}{2(1-\mu)}(-(1-2\mu)c_2+c_2(2-2\mu+\beta z))e^{-\beta z} \quad (73)$$

$$\bar{\sigma}_{zz}(\beta,z)=\frac{-G\beta^2}{2(1-\mu)}(c_2(2-2\mu+\beta z-1+2\mu)e^{-\beta z} \quad (74)$$

$$\bar{\sigma}_{zz}(\beta,z)=\frac{-G\beta^2}{2(1-\mu)}c_2(1+\beta z)e^{-\beta z} \quad (75)$$

$$\bar{\sigma}_{zz}(\beta,z)=\frac{-G\beta^2}{2(1-\mu)}(1+\beta z)e^{-\beta z}\cdot\frac{2(1-\mu)P_0}{\pi\beta^2G} \quad (76)$$

$$\bar{\sigma}_{zz}(\beta,z)=\frac{-P_0}{\pi}(1+\beta z)e^{-\beta z} \quad (77)$$

By inversion, $\sigma_{zz}(x,z)$ is obtained as

$$\sigma_{zz}(x,z)=\int_0^\infty-\frac{P_0}{\pi}(1+\beta z)e^{-\beta z}\cos\beta x\,d\beta \quad (78)$$

$$\sigma_{zz}(x,z)=-\frac{P_0}{\pi}\int_0^\infty(1+\beta z)e^{-\beta z}\cos\beta x\,d\beta \quad (79)$$

$$\sigma_{zz}(x,z)=-\frac{2P_0z^3}{\pi r^4}=-\frac{2P_0z^3}{\pi(x^2+z^2)^2} \quad (80)$$

$$\bar{\tau}_{xz}(\beta,z)=\left\{-\frac{Gi\beta^2}{2(1-\mu)}(-c_1+c_2(1-z\beta))+Gi\beta^2c_2\right\}e^{-\beta z} \quad (81)$$

$$\bar{\tau}_{xz}(\beta,z)=\left\{-\frac{Gi\beta^2}{2(1-\mu)}((1-2\mu)c_2+c_2(1-z\beta))+Gi\beta^2c_2\right\}e^{-\beta z} \quad (82)$$

$$\bar{\tau}_{xz}(\beta,z)=-\frac{Gi\beta^2}{2(1-\mu)}(c_2(1-2\mu+1-z\beta))e^{-\beta z}+Gi\beta^2c_2e^{-\beta z} \quad \ldots(83)$$

$$\bar{\tau}_{xz}(\beta,z) = -\frac{Gi\beta^2}{2(1-\mu)} c_2(2-2\mu - z\beta)e^{-\beta z} + Gi\beta^2 c_2 e^{-\beta z} \quad \ldots(84)$$

$$\bar{\tau}_{xz}(\beta,z) = \left(-\frac{Gi\beta^2}{2(1-\mu)} \frac{2(1-\mu)P_0}{\pi\beta^2 G}(2-2\mu-z\beta) + Gi\beta^2 \frac{2(1-\mu)P_0}{\pi\beta^2 G}\right)e^{-\beta z} \quad \ldots(85)$$

$$\bar{\tau}_{xz}(\beta,z) = \frac{P_0 i}{\pi} z\beta e^{-\beta z} \quad (86)$$

By inversion,

$$\tau_{xz}(x,z) = \int_0^\infty \frac{P_0 i}{\pi} z\beta e^{-\beta z}\cos\beta x\, d\beta \quad (87)$$

$$\tau_{xz}(x,z) = \frac{P_0 iz}{\pi}\int_0^\infty \beta e^{-\beta z}\cos\beta x\, d\beta \quad (88)$$

$$\tau_{xz}(x,z) = \frac{-2P_0 xz^2}{\pi r^2} = \frac{-2P_0 xz^2}{\pi(x^2+z^2)} \quad (89)$$

4.4 Boussinesq-Papkovich Stress Functions in the Physical Domain Variables

By inversion, $\phi(x,z)$ and $\psi(x,z)$ are obtained as follows:

$$\phi(x,z) = \int_0^\infty \left(\frac{2(1-\mu)P_0}{\pi\beta^2 G}\beta z - \frac{2(1-\mu)(1-2\mu)P_0}{\pi\beta^2 G}\right)e^{-\beta z}\cos\beta x\, d\beta \quad (90)$$

$$\phi(x,z) = \frac{2(1-\mu)P_0}{\pi G}\int_0^\infty \frac{1}{\beta^2}(\beta z - (1-2\mu))e^{-\beta z}\cos\beta x\, d\beta \quad (91)$$

$$\phi(x,z) = \frac{2(1-\mu)P_0}{\pi G}\int_0^\infty (\beta z - (1-2\mu))\frac{e^{-\beta z}}{\beta^2}\cos\beta x\, d\beta \quad (92)$$

$$\psi(x,z) = \int_0^\infty \frac{2(1-\mu)P_0}{\pi\beta^2 G}\beta e^{-\beta z}\cos\beta x\, d\beta \quad (93)$$

$$\psi(x,z) = \frac{2(1-\mu)P_0}{\pi G}\int_0^\infty \frac{e^{-\beta z}}{\beta}\cos\beta x\, d\beta \quad (94)$$

4.5 Strain Fields in the Physical Domain Variables

The strain fields are obtained as:

$$\varepsilon_{xx} = \frac{1}{4(1-\mu)}\frac{2(1-\mu)P_0}{\pi G}\int_0^\infty (\beta z - (1-2\mu))e^{-\beta z}\cos\beta x\, d\beta \quad (95)$$

$$\varepsilon_{xx} = \frac{P_0}{2\pi G}\int_0^\infty (\beta z - (1-2\mu))e^{-\beta z}\cos\beta x\, d\beta \quad (96)$$

$$\varepsilon_{xx} = \frac{-P_0}{2\pi G}\int_0^\infty ((1-2\mu) - \beta z)e^{-\beta z}\cos\beta x\, d\beta \quad (97)$$

$$\varepsilon_{xx} = \frac{-P_0}{2\pi G}\left\{(1-2\mu)\int_0^\infty e^{-\beta z}\cos\beta x\, d\beta - \int_0^\infty \beta z\, e^{-\beta z}\cos\beta x\, d\beta\right\} \quad (98)$$

$$\varepsilon_{xx} = \frac{-P_0}{2\pi G}\left\{(1-2\mu)\left(\frac{z}{r^2}\right) - z\left(-\frac{1}{r^2}\right)\left(1-\frac{2z^2}{r^2}\right)\right\} \quad (99)$$

$$\varepsilon_{xx} = \frac{-P_0}{2\pi G}\left\{\frac{(1-2\mu)z}{(x^2+z^2)} + \frac{z(r^2-2z^2)}{(x^2+z^2)^2}\right\}$$

$$\varepsilon_{xx} = \frac{-P_0}{2\pi G}\left\{\frac{(1-2\mu)z}{x^2+z^2} + \frac{z(x^2-z^2)}{(x^2+z^2)^2}\right\} \quad (100)$$

$$\varepsilon_{zz} = \frac{2(1-\mu)P_0}{\pi G}\int_0^\infty -\frac{\beta e^{-\beta z}}{\beta}\cos\beta x\, d\beta - \frac{1}{4(1-\mu)}\cdot\frac{2(1-\mu)P_0}{\pi G}\int_0^\infty \left(\frac{1}{\beta^2}(\beta^3 z - 2\beta^2)e^{-\beta z}\right.$$

$$\left. - (1-2\mu)\beta^2 e^{-\beta z}\right)\cos\beta x\, d\beta \quad (101)$$

$$\varepsilon_{zz} = \frac{2(1-\mu)P_0}{\pi G}\int_0^\infty -e^{-\beta z}\cos\beta x\, d\beta$$

$$-\frac{P_0}{2\pi G}\int_0^\infty ((\beta z - 2) - (1-2\mu))e^{-\beta z}\cos\beta x\, d\beta \quad (102)$$

$$\varepsilon_{zz} = \frac{-P_0}{2\pi G}\int_0^\infty (4(1-\mu) + \beta z - 2 - 1 + 2\mu)e^{-\beta z}\cos\beta x\, d\beta \quad (103)$$

$$\varepsilon_{zz} = \frac{-P_0}{2\pi G}\int_0^\infty (1-2\mu+\beta z)e^{-\beta z}\cos\beta x\, d\beta \quad (104)$$

$$\varepsilon_{zz} = \frac{-P_0}{2\pi G}\left\{(1-2\mu)\int_0^\infty e^{-\beta z}\cos\beta x\, d\beta + z\int_0^\infty \beta e^{-\beta z}\cos\beta x\, d\beta\right\} \quad (105)$$

$$\varepsilon_{zz} = \frac{-P_0}{2\pi G}\left\{(1-2\mu)\cdot\frac{z}{r^2} + z\left(-\frac{1}{r^2}\right)\left(1-\frac{2z^2}{r^2}\right)\right\}$$

$$\varepsilon_{zz} = \frac{-P_0}{2\pi G}\left\{\frac{(1-2\mu)z}{x^2+z^2} - \frac{z(r^2-2z^2)}{(x^2+z^2)^2}\right\} \quad (106)$$

$$\varepsilon_{zz} = \frac{-P_0}{2\pi G}\left\{\frac{(1-2\mu)z}{x^2+z^2} - \frac{z(x^2-z^2)}{(x^2+z^2)^2}\right\} \quad (107)$$

$$\varepsilon_{xz} = \frac{-1}{4(1-\mu)}\cdot\frac{2(1-\mu)P_0}{\pi G}\int_0^\infty -((1-\beta z) + (1-2\mu))e^{-\beta z}\sin\beta x\, d\beta$$

$$+\frac{1}{2}\cdot\frac{2(1-\mu)P_0}{\pi G}\int_0^\infty \frac{-\beta e^{-\beta z}}{\beta}\sin\beta x\, d\beta \quad (108)$$

$$\varepsilon_{xz} = \frac{P_0}{2\pi G}\int_0^\infty (1-\beta z+1-2\mu)e^{-\beta z}\sin\beta x\, d\beta + \frac{(1-\mu)P_0}{\pi G}\int_0^\infty -e^{-\beta z}\sin\beta x\, d\beta \quad \ldots(109)$$

$$\varepsilon_{xz} = \frac{P_0}{2\pi G}\int_0^\infty (2-2\mu-\beta z)e^{-\beta z}\sin\beta x\, d\beta + \frac{2(1-\mu)P_0}{2\pi G}\int_0^\infty -e^{-\beta z}\sin\beta x\, d\beta \quad \ldots(110)$$

$$\varepsilon_{xz} = \frac{P_0}{2\pi G}\int_0^\infty (2-2\mu-\beta z - 2(1-\mu))e^{-\beta z}\sin\beta x\, d\beta \quad (111)$$

$$\varepsilon_{xz} = \frac{P_0}{2\pi G}\int_0^\infty -\beta z\, e^{-\beta z}\sin\beta x\, d\beta \quad (112)$$

$$\varepsilon_{xz} = \frac{-P_0}{2\pi G}\int_0^\infty \beta z\, e^{-\beta z}\sin\beta x\, d\beta \quad (113)$$

$$\varepsilon_{xz} = \frac{-P_0 z}{2\pi G}\int_0^\infty \beta e^{-\beta z}\sin\beta x\, d\beta \quad (114)$$

$$\varepsilon_{xz} = \frac{-P_0 z}{2\pi G}\frac{\partial}{\partial z}\left(\frac{x}{r^2}\right) = \frac{-P_0 z}{2\pi G}\frac{2xz}{r^4} \quad (115)$$

$$\varepsilon_{xz} = \frac{-P_0}{\pi G}\frac{xz^2}{r^4} = \frac{-P_0 xz^2}{\pi G(x^2+z^2)^2} \quad (116)$$

4.6 Displacement Fields

The displacement fields are obtained by integrating the strain-displacement relations. Thus,

$$\int_0^x \frac{\partial u_x}{\partial x}dx = \int_0^x \varepsilon_{xx}\partial x = \int \partial u_x \quad (117)$$

$$u_{xx}(x,z) - u_x(x=0,z) = \int_0^x \varepsilon_{xx}(t,x)dt \quad (118)$$

$$= \frac{-P_0}{2\pi G}\int_0^x\int_0^\infty ((1-2\mu)-\beta z)e^{-\beta z}\cos\beta x\, d\beta dx \qquad (119)$$

$$= \frac{-P_0}{2\pi G}\int_0^\infty ((1-2\mu)-\beta z)\frac{e^{-\beta z}}{\beta}\sin\beta x\, d\beta \qquad (120)$$

$$= \frac{-P_0}{2\pi G}\left\{(1-2\mu)\int_0^\infty \frac{e^{-\beta z}}{\beta}\sin\beta x\, d\beta - \int_0^\infty z\, e^{-\beta z}\sin\beta x\, d\beta\right\} \qquad (121)$$

$$= \frac{-P_0}{2\pi G}\left\{(1-2\mu)\tan^{-1}\frac{x}{z} - \frac{zx}{r^2}\right\} \qquad (122)$$

$$= \frac{-P_0}{2\pi G}\left\{(1-2\mu)\tan^{-1}\left(\frac{x}{z}\right) - \frac{xz}{x^2+z^2}\right\} \qquad (123)$$

Similarly,

$$\int_0^z \frac{\partial u_z}{\partial z}dz = \int_0^z \varepsilon_{zz}\partial z = \int_0^z \partial u_z = \int_0^z \partial w \qquad (124)$$

$$u_z(x,z) - u_z(x, z=0) = \int_0^z \varepsilon_{zz}\, dz \qquad (125)$$

$$u_z(x,z) - u_z(x,z=0) = \frac{-P_0}{2\pi G}\int_0^z\int_0^\infty (1-2\mu+\beta z)e^{-\beta z}\cos\beta x\, d\beta dz \qquad (126)$$

$$u_z(x,z) - u_z(x,z=0) = \frac{-P_0}{2\pi G}\left\{(1-2\mu)\int_0^z\int_0^\infty e^{-\beta z}\cos\beta x\, d\beta dz + \int_0^z\int_0^\infty \beta z e^{-\beta z}\cos\beta x\, d\beta dz\right\} \qquad (127)$$

$$u_z(x,z) - u_z(x,z=0) = \frac{-P_0}{2\pi G}\left\{-(1-2\mu)\int_0^\infty\left(\frac{1-e^{-\beta z}}{\beta}\right)\cos\beta x\, d\beta + \int_0^\infty \frac{1}{\beta}\left((1-(1+\beta z)e^{-\beta z}\right)\cos\beta x\, d\beta\right\} \qquad (128)$$

$$u_z(x,z) - u_z(x,z=0) = \frac{-P_0}{2\pi G}\left\{(1-2\mu)\tan^{-1}\frac{x}{z} - \frac{xz}{(x^2+z^2)}\right\} \qquad (129)$$

5 Discussion

In this work, the Fourier cosine transform method was used successfully to solve the elasticity problem of point load P_0 applied vertically at the origin O of an elastic half plane. The elastic half plane material was considered homogeneous, isotropic and linear elastic. The Boussinesq-Papkovich displacement potential functions, defined as Equations (3) and (4) were used to express the strain fields as Equations (9), (10), and (13) using the strain-displacement relations of infinitesimal elasticity theory. The stress fields were obtained in terms of the Boussinesq-Papkovich potential functions, using the stress-strain relations, as Equations (22) – (24). The Fourier cosine transformation of the biharmonic stress compatibility equation was used to obtain the ordinary differential equation in Fourier cosine transform space which suitable Boussinesq-Papkovich potential functions must satisfy as Equation (30). The bounded solutions for the Bousinesq-Papkovich potential functions were obtained as Equations (35) and (36). This yielded the Bousinesq-Papkovich potential functions as Equations (35) and (36); in the Fourier cosine transform space. The Boussinesq-Papkovich potential functions in the physical domain space were obtained by inversion as Equations (38) and (40). The stresses were obtained in the Fourier cosine transform space as Equations (46), (50) and (54). The boundary conditions for shear stresses to vanish on the boundary z = 0, was used to obtain the relationship between the unknown constants as Equation (57). The boundary condition on the equilibrium of internal vertical stress and applied point load at the origin was used to find the second unknown constant as Equation (63). All the constants were thus determined. The stress fields in the transform space are fully determined in terms of the point load and the elastic properties of the half plane as Equations (67), (77) and (86). By inversion, the stress-fields were obtained in the physical domain variable as Equations (70), (80) and (89). The strain fields were obtained in the physical domain variables as Equations (100), (107) and (116). The displacement fields were obtained by integration of the strain fields as Equations (123), and (129). It was observed that the expressions obtained for the stresses and displacements fields were identical to the solutions found in the technical literature on the subject.

6 Conclusions

The conclusions of the work are as follows:

(i) the Fourier cosine transform method is an effective mathematical tool for solving the elasticity problem of a point load applied vertically at the origin of an isotropic, homogeneous elastic half plane.

(ii) the Boussinesq-Papkovich displacement potential function formulation of the problem simplified the mathematical solution of the boundary value problem of elasticity to the determination of suitable functions of the two dimensional coordinate variables that solve the biharmonic stress compatibility equation, subject to the traction and boundary conditions, and also satisfied the boundedness requirements for stresses and displacements.

(iii) the application of the Fourier cosine transformation simplified the biharmonic stress compatibility equation to a linear fourth order ordinary differential equation with constant coefficients; which is more readily amenable to closed form mathematical solutions than the original biharmonic partial differential equation.

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